

Fully Open Extensions to the D'Hondt Method

Luis Sánchez-Fernández
Dept. Telematic Engineering,
Universidad Carlos III de Madrid,
E-28911 Leganés, Spain

Norberto Fernández García
Centro Universitario de la Defensa,
Escuela Naval Militar,
E-36920 Marín, Spain

Jesús A. Fisteus
Dept. Telematic Engineering,
Universidad Carlos III de Madrid,
E-28911 Leganés, Spain

September 20, 2016

Abstract

In this study, we propose two voting rules that extend the D'Hondt method for approval-based multi-winner elections. Our theoretical results prove that one of such rules preserves the main properties of the original D'Hondt method: house monotonicity, lower quota satisfaction, and population monotonicity, and that it can be executed in polynomial time.

1 Introduction

Decision making based on the aggregation of possibly conflicting preferences is a central problem in the field of social choice that has received a considerable amount of attention from the artificial intelligence research community [13, 42, 8, 19, 20, 2, 44]. A voting system is the usual way of making collective decisions.

The selection of a single candidate out of several is the most common scenario in which voting systems are studied. However, the scenario in which a winning set of candidates is selected (multi-winner elections) is also frequent. The most typical situation is the election of a parliament or a committee. Multi-winner elections can also be used by software agents in scenarios such as deciding on a set of plans [19] or resource allocation [44].

One of the basic characteristics of a multi-winner voting rule is the way in which agents (voters) cast their votes. Two alternatives are commonly used: 1) ranked ballots, in which agents have to provide a total order of the candidates; and 2) approval ballots, in which agents simply approve as many candidates as they like. One of the advantages of approval ballots compared to ranked ballots is the simplicity of the ballots [2].

Multi-winner voting rules are applied often in scenarios in which it is desirable that the set of winners represents the different opinions or preferences of the agents involved in the election. However, in the case of approval-based ballots, there is a lack of multi-winner voting rules that capture well the idea of representation and that, at the same time, can be computed efficiently (that is, in polynomial time). An initial study of this problem was presented by Aziz *et al.* [2] and it has been recently extended by Sánchez Fernández *et al.* [41]. The problem of finding sets of winners that represent the different opinions or preferences of the agents is related but different to the problem of minimization of the agent dissatisfaction with respect to a set of winners as a whole [10, 6].

In contrast, the problem of achieving proportional representation has been quite satisfactorily solved in the case of party-list proportional representation systems [18, 36, 21, 24]. In party-list proportional representation systems candidates are organized in lists (it is usually assumed that the lists of candidates are disjoint). The most typical situation is the use of *closed lists*. When closed lists are used, each voter has to choose one from several ordered lists of candidates. Each list is then assigned a number of seats based on the votes that it has received. The elected candidates from each list are those ranked first in the list until the full number of seats has been assigned. Therefore, the agents have no direct choice about *who* finally represents them.

Several voting systems have also been proposed that are based on party-

list proportional representation systems but that *open* the way in which voters cast their votes. In some cases, the voters are still restricted to voting for a single list, but they may influence the order of the candidates within the list that they have selected. Finally, in the most open party-list proportional representation systems, the voters have total freedom to choose the candidates who they would like to support even if they belong to different lists.

As we have already said, there are several party-list proportional representation systems that provide reasonable outcomes from the point of view of achieving proportional representation. One of the most widely used is the D'Hondt method (in the USA, this is also known as Jefferson's method), which is used to elect the parliaments in many countries such as Austria, Belgium, Spain, Netherlands, Luxembourg, Switzerland, and Israel.

The main disadvantage of party-list proportional representation systems relies precisely in the need to use lists of candidates: in many elections the use of lists of candidates may not be desirable or simply they are not available. In the case of multi-agent systems and other scenarios in artificial intelligence in which voting systems can be used the usual situation is that lists of candidates do not exist and therefore party-list proportional representation systems cannot be used.

In this study, we propose two voting rules that extend the D'Hondt method in a scenario where: 1) lists of candidates are not available; and 2) approval ballots are used. We call such extensions fully open extensions to the D'Hondt method. The difference between both approaches is that the first one studied is an iterative algorithm that makes local optimizations at each iteration while the second voting rule follows the same ideas that the first one except that it makes a global optimization. It should not be surprising that the first voting rule can be computed in polynomial time while the second it can not. We share the same view of Elkind *et al.* [20] and Caragiannis *et al.* [11] that consider iterative versions of voting rules that make global optimizations as full-fledged multi-winner voting rules. We will study first the iterative version and we will show that it satisfies (adjusted versions of) the properties of the D'Hondt method: house monotonicity, lower quota satisfaction, and population monotonicity. The version that makes a global optimization will be studied later together with other voting rules already available in the state of the art.

The remainder of this paper is organized as follows. Section 2 provides a summary of the main notations and symbols used throughout this study. Section 3 reviews the D'Hondt method and its main properties. Section 4 defines the requirements that should be fulfilled by an extension of the D'Hondt method for approval ballots. Section 5 introduces the concept of *support* used in the development of the proposed extension. Section 6 describes the pro-

posed (iterative) extension and an example is presented. Section 7 provides proofs to demonstrate that the proposed extension fulfills all the requirements defined in section 4. The last two sections, 8 and 9, compare the proposed extension with other available voting systems (including the other rule proposed in this paper), and some concluding remarks are also given.

2 Notation

Throughout this study, the following mathematical notations are used:

- For any given set \mathcal{A} , $|\mathcal{A}|$ represents its cardinality.
- For any given set \mathcal{A} , $2^{\mathcal{A}}$ represents its power set.
- For any real number x , $\lfloor x \rfloor$ is the largest integer that is less than or equal to x and $\lceil x \rceil$ is the smallest integer that is greater than or equal to x .
- The maximum of a finite and non-empty set \mathcal{A} of real numbers (represented as $\max \mathcal{A}$) is the element of \mathcal{A} strictly greater than all other elements of \mathcal{A} (we assume that sets cannot contain duplicates).
- The supremum of a (possibly infinite) bounded from above and non-empty set \mathcal{A} of real numbers (represented as $\sup \mathcal{A}$) is the smallest real number that it is greater than or equal to all elements of \mathcal{A} .

Observe that the supremum of a set \mathcal{A} may not belong to \mathcal{A} . For instance, let \mathcal{I} be the set of all the real numbers that belong to the open interval $(0, 1)$. Then, $\sup \mathcal{I} = 1$, and $1 \notin \mathcal{I}$.

On the other hand, if \mathcal{A} is a finite and non-empty set of real numbers it is always $\max \mathcal{A} = \sup \mathcal{A}$.

- The minimum of a finite and non-empty set \mathcal{A} of real numbers (represented as $\min \mathcal{A}$) is the element of \mathcal{A} strictly smaller than all other elements of \mathcal{A} .
- Given a finite and non-empty set \mathcal{A} and a function f from \mathcal{A} to a set of integer or real numbers, we represent the element of \mathcal{A} that maximizes (respectively minimizes) f by $\operatorname{argmax}_{\mathcal{A}} f$ (respectively $\operatorname{argmin}_{\mathcal{A}} f$).

$$\begin{aligned} \operatorname{argmax}_{\mathcal{A}} f &= x \Leftrightarrow x \in \mathcal{A}, \text{ and } \forall y \in \mathcal{A}, f(x) \geq f(y) \\ \operatorname{argmin}_{\mathcal{A}} f &= x \Leftrightarrow x \in \mathcal{A}, \text{ and } \forall y \in \mathcal{A}, f(x) \leq f(y) \end{aligned}$$

The following notations related to elections are also used:

- \mathcal{V} represents the set of agents (voters) involved in the election.
- \mathcal{C} represents the set of candidates.
- S represents the total number of seats that must be allocated in the election. We assume that $|\mathcal{C}| \geq S \geq 1$.

We will use the symbol σ to refer to an election. We will use several election examples along the paper. Each example will be identified with a letter superscript like σ^a , σ^b , etc. Sometimes we will use a letter and a number when we have similar or related elections like σ^{a1} .

When we have to refer to several elections (in general) in definitions, theorems, and proofs we will use number subscripts (like σ_1 , σ_2 , etc.) or neither superscript nor subscript (only σ).

3 Review of the D'Hondt method for closed lists

The D'Hondt method is a particular example from a family of voting systems known as highest averages methods or divisor methods [21]. The use of candidate lists is required. To assign S seats to the lists, the votes obtained by each list are divided by a sequence of divisors. S seats are then assigned to the lists with the highest S quotients.

The different highest averages methods use the same procedure described above but they vary in the particular sequence of divisors. In the case of the D'Hondt method, the sequence $1, 2, 3, 4, \dots$ is used.

An example of the use of the D'Hondt method is shown in Table 1, where five seats must be assigned to three lists: **A** (composed of candidates a_1, \dots, a_5), **B** (composed of candidates b_1, \dots, b_5), and **C** (composed of candidates c_1, \dots, c_5), which have received 5,100; 3,150; and 1,750 votes, respectively. As shown in Table 1, the D'Hondt method would assign three seats to list **A**, one to **B**, and one to **C**.

As discussed earlier, if closed lists are used, the elected candidates from each list are those ranked first in each list until the required number of seats has been assigned. So, for this example, the elected candidates would be a_1, a_2, a_3, b_1 and c_1 .

Lists		A	B	C
Votes		5,100	3,150	1,750
Divisors	1	5,100.0	3,150.0	1,750.0
	2	2,550.0	1,575.0	875.0
	3	1,700.0	1,050.0	583.3
	4	1,275.0	787.5	437.5
	5	1,020.0	630.0	350.0
Seats won		3	1	1

Table 1: Example of the use of the D'Hondt method

3.1 Formal model

A closed lists election can be specified by a 5-tuple, $\sigma = \langle \mathcal{V}, \mathcal{C}, S, \mathcal{L}, B \rangle$, where \mathcal{V}, \mathcal{C} and S have been defined in section 2, and:

- \mathcal{L} is the set of lists. For each list $\ell_i \in \mathcal{L}$, an ordered set of candidates $\mathcal{C}_i \subseteq \mathcal{C}$ is defined that comprises the list. The different \mathcal{C}_i are disjoint from each other. Usually, it is also assumed that for each list $\ell_i \in \mathcal{L}$, it holds that $|\mathcal{C}_i| \geq S$.
- B is a function $B : \mathcal{L} \rightarrow \mathbb{N}$ that represents the number of votes obtained by each list (election result).

The sequence of divisors used by a specific divisor method can be represented as a function $d : \mathbb{N} \rightarrow \mathbb{R}^+ \cup \{0\}$ ¹ such that $d(i) < d(i+1)$ for all i (in the case of the D'Hondt method, $d(i) = i+1$).

The computation of the winners using a divisor method requires two steps. In the first step, the number of seats allocated to each list is determined. In the second step, for each list, the candidates ranked first in the list are allocated until the number of seats that needs to be selected has been assigned from the list.

The allocation of seats to lists requires S iterations. At each iteration, one seat is allocated to one of the lists. Let $a(j, \ell)$ be the number of seats allocated to list ℓ at iteration j (assuming that $a(0, \ell) = 0$ for all ℓ). The score $s(j, \ell)$ for each list ℓ at iteration j is computed as follows:

$$s(j, \ell) = \frac{B(\ell)}{d(a((j-1), \ell))}. \quad (1)$$

¹0 is used as a divisor in the Equal Proportions method [18] to guarantee that each list receives at least one representative. It is used in the USA to allocate seats of the House of Representatives to states.

At each iteration, a seat is then allocated to the list with the highest score.

Let $DH(\sigma) \subseteq \mathcal{C}$ be the set of S candidates elected for election σ using the D'Hondt method.

3.2 Properties of the D'Hondt method

It is well known [18, 4] that the D'Hondt method possesses the following properties.

- **House Monotonicity.** If the number of seats is increased from S to $S + 1$, house monotonicity guarantees that no list will be penalized by receiving fewer seats.

Formally, for any pair of elections, $\sigma = \langle \mathcal{V}, \mathcal{C}, S, \mathcal{L}, B \rangle$ and $\sigma_1 = \langle \mathcal{V}, \mathcal{C}, (S + 1), \mathcal{L}, B \rangle$, it must be true that $DH(\sigma) \subset DH(\sigma_1)$.

- **Lower Quota Satisfaction.** A party-list proportional representation system should aim to assign the exact fraction of seats required from each list, i.e., the so-called quota:

$$q_i = \frac{B(\ell_i)}{|\mathcal{V}|} S. \quad (2)$$

Lower quota satisfaction guarantees that each list must receive a minimum number of seats of $\lfloor q_i \rfloor$.

Formally, for any election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, \mathcal{L}, B \rangle$, and for any list, $\ell_i \in \mathcal{L}$, this must be $|DH(\sigma) \cap \mathcal{C}_i| \geq \lfloor q_i \rfloor$.

- **Population Monotonicity.** If a list increases its number of votes whereas the votes of all the other lists remain the same, population monotonicity guarantees that such list will not be penalized by receiving fewer seats.

Formally, for any election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, \mathcal{L}, B \rangle$, and any list $\ell_i \in \mathcal{L}$, if another election $\sigma_1 = \langle \mathcal{V}', \mathcal{C}, S, \mathcal{L}, B_1 \rangle$ fulfills:

$$B_1(\ell_i) > B(\ell_i) \quad (3)$$

$$\forall \ell_j \in \mathcal{L}, \ell_j \neq \ell_i \Rightarrow B_1(\ell_j) = B(\ell_j), \quad (4)$$

then it must be true that $DH(\sigma) \cap \mathcal{C}_i \subseteq DH(\sigma_1)$.

4 Formal model and requirements of the extension of the D'Hondt method for approval ballots

In the context of voting systems in general, several research studies exist (probably the most famous is the Arrow's Theorem [1]) that state that it is not possible to develop a “perfect” voting system. Along the same idea, we believe that it is impossible to prove that certain approval-based multi-winner voting rule is the “best” extension to the D'Hondt method. An extension can be better than other in certain aspects while the second can be better than the first in others. We will come back later to this issue in section 8, in which we compare the voting rules we present in this paper with other approval-based multi-winner voting rules that could also be considered as extensions to the D'Hondt method. However, it is possible to define a set of requirements that a “good” extension to the D'Hondt method must fulfill.

An obvious requirement is that when the agents behave as if a closed list ballot structure is used, then the assignment of seats to candidates should be the same as that with the original D'Hondt method.

It also appears to be reasonable that such extensions share the desirable social choice properties of the original D'Hondt method: house monotonicity, lower quota satisfaction, and population monotonicity. However, since these properties are defined in a context where closed lists are used, then it is necessary to extend their definitions. As discussed by Bartholdi III *et al.* [5], computational intractability is a major consideration when adopting a voting system. Therefore, an additional requirement is that the voting rule should be computable in polynomial time.

In the remainder of this section, we formalize the aforementioned requirements.

4.1 Formal model for approval-based multi-winner elections

We specify an approval-based multi-winner election by a 4-tuple, $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, where the election result is modeled as a function B from the power set of \mathcal{C} to natural numbers. For any subset \mathcal{A} of \mathcal{C} , the number of agents that approve every candidate in \mathcal{A} and not any other candidate is $B(\mathcal{A})$.

$$B : 2^{\mathcal{C}} \rightarrow \mathbb{N} \tag{5}$$

Let M be a voting rule for approval-based multi-winner elections. Then,

$M(\sigma) \subseteq \mathcal{C}$ is the set of S candidates elected for election σ using the voting rule M .

4.2 Example

Suppose that an election $\sigma^a = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ is conducted to allocate three seats among seven candidates $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}\}$. The total number of agents is 45,000 (although only 43,000 cast valid votes). Table 2 summarizes the values of the components of σ^a . The result of the election is shown in Table 3 (for any candidate subset \mathcal{A} not shown in Table 3, it is assumed that $B(\mathcal{A}) = 0$).

Component of σ^a	Value
$ \mathcal{V} $	45,000
\mathcal{C}	$\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}\}$
S	3
B	See table 3

Table 2: Components of election σ^a

Candidates	Number of votes
$\{\mathbf{a}, \mathbf{b}\}$	10,000
$\{\mathbf{a}, \mathbf{c}\}$	6,000
$\{\mathbf{b}\}$	4,000
$\{\mathbf{c}\}$	5,500
$\{\mathbf{d}\}$	9,500
$\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$	5,000
$\{\mathbf{e}\}$	3,000

Table 3: Result of election σ^a

4.3 Requirements for an extension of the D’Hondt method

As discussed at the beginning of this section, the following properties are desirable for an extension of the D’Hondt method for approval-based multi-winner elections.

- **Equivalent to D’Hondt under closed lists.** Agents in an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ vote as if closed lists are used when a set of candidate subsets $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ exists, for which it holds that:

$$|\mathcal{V}| > 0 \quad (6)$$

$$\forall i = 1, \dots, n; \mathcal{C}_i \subseteq \mathcal{C} \quad (7)$$

$$\forall i = 1, \dots, n; |\mathcal{C}_i| \geq S \quad (8)$$

$$\forall i, j = 1, \dots, n; i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset \quad (9)$$

$$\forall \mathcal{C}' \in 2^{\mathcal{C}}, \mathcal{C}' \notin \{\mathcal{C}_1, \dots, \mathcal{C}_n\} \Rightarrow B(\mathcal{C}') = 0. \quad (10)$$

If this holds, then σ is equivalent to the closed list election $\sigma_1 = \langle \mathcal{V}, \mathcal{C}_c, S, \mathcal{L}, B_1 \rangle$, where:

$$\mathcal{L} = \{\mathcal{C}_1, \dots, \mathcal{C}_n\} \quad (11)$$

$$\mathcal{C}_c = \bigcup_{i=1}^n \mathcal{C}_i \quad (12)$$

$$\forall i = 1, \dots, n; B_1(\mathcal{C}_i) = B(\mathcal{C}_i). \quad (13)$$

In addition, for each 'list' \mathcal{C}_i , a specific ordering of the candidates must be provided. Then, with a voting rule for approval-based multi-winner elections M that is equivalent to D'Hondt under closed lists, it must hold that $M(\sigma)$ can be equal $DH(\sigma_1)$ (subject to how the ties between candidates are broken).

- **House Monotonicity.** The definition of this property is essentially the same as that used for closed lists.

A voting rule M for approval-based multi-winner elections is house monotonic if for any pair of elections $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and $\sigma_1 = \langle \mathcal{V}, \mathcal{C}, (S+1), B \rangle$, it holds that $M(\sigma) \subset M(\sigma_1)$.

- **Lower Quota Satisfaction.** An initial description of this property can be expressed in the following terms. If a number of agents that is sufficiently large to satisfy a given quota q agrees to vote for the same set of candidates, then at least $\lfloor q \rfloor$ of these candidates must be elected. It should be noted that this property is similar to the corresponding property for closed lists, because voting for a list is equivalent to voting for all of the candidates in the list.

We can then strengthen this property by allowing that each agent can approve additional candidates not included in the common set of candidates.

Formally, for any given election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, a candidate subset $\mathcal{A} \subseteq \mathcal{C}$, and a set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ such that:

$$\mathcal{A} \subseteq y_j, \forall j = 1, \dots, n \quad (14)$$

$$q = \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{V}|} S \quad (15)$$

$$|\mathcal{A}| \geq \lfloor q \rfloor \quad (16)$$

a voting rule M for approval-based multi-winner elections that satisfies the lower quota must fulfill $|M(\sigma) \cap \bigcup_{j=1}^n y_j| \geq \lfloor q \rfloor$.

- **Population Monotonicity.** The original condition for closed lists is now translated into two conditions by considering the two different ways in which a candidate can receive additional votes without changing the votes for the other candidates: a) because an agent decides to vote for a candidate *in addition* to the other candidates that the agent has already selected, and b) from *new* agents that vote *only* for the candidate.

We can also strengthen this property by allowing that a subset of the elected candidates (instead of a single candidate) increase their votes in any of the two ways we have described in the previous paragraph.

Formally, a voting rule M for approval-based multi-winner elections is population monotonic if for any $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and for any non-empty candidate subset $\mathcal{G} \subseteq M(\sigma)$, the following conditions hold.

1. For any $\mathcal{A} \subseteq \mathcal{C}$, such that $\mathcal{G} \cap \mathcal{A} = \emptyset$ and $B(\mathcal{A}) \geq 1$, consider the election $\sigma_1 = \langle \mathcal{V}_1, \mathcal{C}, S, B_1 \rangle$, where

$$|\mathcal{V}_1| = |\mathcal{V}| \quad (17)$$

$$B_1(\mathcal{A}) = B(\mathcal{A}) - 1 \quad (18)$$

$$B_1(\mathcal{A} \cup \mathcal{G}) = B(\mathcal{A} \cup \mathcal{G}) + 1 \quad (19)$$

$$\forall \mathcal{X} \in 2^{\mathcal{C}}, \mathcal{X} \neq \mathcal{A}, \mathcal{X} \neq (\mathcal{A} \cup \mathcal{G}) \Rightarrow B_1(\mathcal{X}) = B(\mathcal{X}), \quad (20)$$

and thus it must hold that $\mathcal{G} \cap M(\sigma_1) \neq \emptyset$.

2. Consider the election $\sigma_2 = \langle \mathcal{V}_2, \mathcal{C}, S, B_2 \rangle$, where

$$|\mathcal{V}_2| = |\mathcal{V}| + 1 \quad (21)$$

$$B_2(\mathcal{G}) = B(\mathcal{G}) + 1 \quad (22)$$

$$\forall \mathcal{X} \in 2^c, \mathcal{X} \neq \mathcal{G} \Rightarrow B_2(\mathcal{X}) = B(\mathcal{X}), \quad (23)$$

and thus it must hold that $\mathcal{G} \cap M(\sigma_2) \neq \emptyset$.

4.3.1 A note on the properties

The lower quota property was first proposed by Sánchez Fernández *et al.* in [41] with the name of “Proportional Justified Representation”. Our definitions for house monotonicity and population monotonicity are similar to related properties for ranked ballots proposed by Elkind *et al.* [20] and Woodall [46].

5 A Formal Model for the Concept of Support

Before presenting the extension proposed in this study to the D’Hondt method, it is necessary to introduce and formalize the concept of *support*.

One way of viewing the D’Hondt method is to consider that a vote received by each list is distributed among the candidates elected from that list. For instance, from this viewpoint, in the example shown in Table 1, each of the candidates elected from list **A** is supported by 1,700 agents, the candidate elected from list **B** is supported by 3,150 agents, and the candidate elected from list **C** is supported by 1,750 agents. It should be noted that the set of candidates selected by the D’Hondt method is optimal in the sense that it maximizes the support received by the least supported candidate [18]. For instance, if another method selects two candidates from list **B**, each of them would be supported by 1,575 agents, which is less than the minimum support obtained with D’Hondt, i.e., 1,700 agents for each of the candidates selected from list **A**.

In the same manner, when approval ballots are used, the votes (even fractions of *each vote*) can be distributed among the candidates elected by the agents. It should be noted that there may be many different ways to distribute the votes, thereby leading to different support values for each candidate. In this section, we present a formal model for support without choosing any particular way of distributing the votes.

For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, we define the *support distribution functions* family $\mathfrak{F}_{\sigma, \mathcal{A}}$ as the set of all functions that can be used to distribute votes among the candidates in \mathcal{A} .

$\mathfrak{F}_{\sigma, \mathcal{A}}$ is defined as the set of all functions F from $(2^{\mathcal{C}} \times \mathcal{A})$ to \mathbb{R} that satisfy the following conditions.

$$F(y, c) \geq 0 \quad \text{for each } y \in 2^{\mathcal{C}} \text{ and for each } c \in \mathcal{A} \quad (24)$$

$$F(y, c) = 0 \quad \text{for each } y \in 2^{\mathcal{C}} \text{ and for each } c \in \mathcal{A} \text{ such that } c \notin y \quad (25)$$

$$\sum_{c \in \mathcal{A} \cap y} F(y, c) = B(y) \quad \text{for each } y \text{ such that } y \cap \mathcal{A} \neq \emptyset \quad (26)$$

For each candidate subset $y \in 2^{\mathcal{C}}$, $F(y, c)$ is the fraction of all the agent votes that approve only the candidates in y , that F assigns to candidate c . Equation (24) states that candidates cannot receive negative shares of votes. Equation (25) states that a candidate that has not been approved by an agent cannot receive any positive support from them. Finally, Equation (26) states that all votes in support of any of the candidates in \mathcal{A} must be distributed.

Observe that, as a consequence of Equation (25) when $y \cap \mathcal{A} = \emptyset$ it has to be $F(y, c) = 0$ for all candidates in \mathcal{A} . That is, an agent that does not approve any candidate in \mathcal{A} does not give any support to such candidates.

Therefore, based on the example in Table 3 and Equations (25) and (26), any support distribution function $F \in \mathfrak{F}_{\sigma^a, \mathcal{A}^a}$, where $\mathcal{A}^a = \{\mathbf{a}, \mathbf{c}, \mathbf{e}\}$, must fulfill the following conditions.

$$\begin{aligned} F(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) &= 10,000 \\ F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) + F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) &= 6,000 \\ F(\{\mathbf{c}\}, \mathbf{c}) &= 5,500 \\ F(\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}, \mathbf{e}) &= 5,000 \\ F(\{\mathbf{e}\}, \mathbf{e}) &= 3,000 \\ F(y, c) &= 0 \text{ for each other } (y, c) \end{aligned}$$

Given a support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}$, we can compute the support $Supp_F$ received by each candidate $c \in \mathcal{A}$ under the support distribution function as follows.

$$Supp_F(c) = \sum_{y \in 2^{\mathcal{C}}} F(y, c) \quad (27)$$

For instance, the function $F^a \in \mathfrak{F}_{\sigma^a, \mathcal{A}^a}$ characterized by $F^a(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) = 2,222.22$ (which implies that $F^a(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) = 3,777.78$) leads to the following values of $Supp_{F^a}$.

$$\begin{aligned} Supp_{F^a}(\mathbf{a}) &= 12,222.22 \\ Supp_{F^a}(\mathbf{c}) &= 9,277.78 \\ Supp_{F^a}(\mathbf{e}) &= 8,000.00 \end{aligned}$$

6 Extended D'Hondt method

The method proposed in the present study follows a simple strategy: the selection of the elected candidates is performed by an iterative algorithm, where at each iteration, the unelected candidate with greatest support must be chosen. However, as indicated in the previous section, there are many different ways of distributing the support of agents among a set of candidates.

The electoral method proposed in the present study is based on the same principle as the D'Hondt method, where the support for each candidate is computed using a support distribution function that maximizes the support for the least supported elected candidate. While the requirements presented in section 4.3 are generally desirable for approval-based multi-winner voting systems, it is not trivial to devise a voting rule that fulfills all of them. In fact, to the best of our knowledge, none of the existing approval-based multi-winner voting systems satisfy all of the requirements presented in section 4.3. By contrast, our extension to the D'Hondt method follows the principle of maximizing the support for the least supported elected candidate but, as proved in section 7, it also satisfies all of the requirements in section 4.3.

For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, let $\max\text{Min}(\sigma, \mathcal{A})$ be the supremum of the support for all possible support distribution functions in $\mathfrak{F}_{\sigma, \mathcal{A}}$ for the least supported candidate in \mathcal{A} .

$$\max\text{Min}(\sigma, \mathcal{A}) = \sup_{\forall F \in \mathfrak{F}_{\sigma, \mathcal{A}}} \min_{\forall c \in \mathcal{A}} Supp_F(c) \quad (28)$$

Observe that we have had to use the supremum (instead of the maximum) because for a given election σ and candidate subset \mathcal{A} there may be infinite support distribution functions in $\mathfrak{F}_{\sigma, \mathcal{A}}$. Election σ^a and candidate subset \mathcal{A}^a are an example of this.

As discussed in section 2, it is not always true that the supremum of a given infinite set \mathcal{X} belongs to \mathcal{X} . However, Lemma 1 states the existence

of a support distribution function for which the support obtained by the least supported candidate is equal to $\max\text{Min}(\sigma, \mathcal{A})$ for any approval-based multi-winner election σ and any non-empty candidate subset \mathcal{A} .

LEMMA 1 *For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, a support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}$ exists such that:*

$$\min_{\forall c \in \mathcal{A}} \text{Supp}_F(c) = \max\text{Min}(\sigma, \mathcal{A}).$$

Proof.

First, we show that the requirements that must be fulfilled by F can be represented as a linear programming problem [15, 16, 37, 7]. A well-known result in linear programming (e.g., see [7], page 19) shows that if a linear programming problem is feasible and bounded, then it has a finite optimal solution. We show that this is the case for the linear program that we define, and that the existence of F follows from this result.

1. A linear programming problem can be expressed in the following form:

$$\begin{aligned} & \text{maximize} && d^T x \\ & \text{subject to} && A_1 x = b_1 \\ & \text{and} && A_2 x \geq b_2 \\ & \text{and} && x \geq 0 \\ & , \end{aligned}$$

where:

- x is a column vector of dimension n , where n is the total number of variables of the problem.
- d^T is a row vector, which is also of dimension n , and $d^T x$ is the goal or objective function of the linear programming problem.
- A_1 is a matrix of dimension $m_1 \times n$, b_1 is a column vector of dimension m_1 , A_2 is a matrix of dimension $m_2 \times n$, and b_2 is a column vector of dimension m_2 . $A_1 x = b_1$ and $A_2 x \geq b_2$ are the constraints on the linear programming problem. The total number of equations in the constraints is $m = m_1 + m_2$.

We need to represent the requirements that must be fulfilled by F as a linear programming problem. Initially, for each pair $(y, c) \in 2^{\mathcal{C}} \times \mathcal{A}$ the value of F for (y, c) is a variable in the linear program.

The requirements that a support distribution function must fulfill, defined in Equations (24) to (26) in section 5 ($F(y, c) \geq 0$ for each (y, c) , $F(y, c) = 0$ for each (y, c) such that $c \notin y$, and $\sum_{c \in y \cap \mathcal{A}} F(y, c) = B(y)$ for each y such that $y \cap \mathcal{A} \neq \emptyset$), can be expressed in the following terms. The first requirement is satisfied by the fact that all the variables in the linear program must take values greater than or equal to zero. The variables $F(y, c)$ for which $c \notin y$ can be omitted from the linear programming problem because for such variables it is $F(y, c) = 0$ due to the second requirement. Also, for those y such that $B(y) = 0$ we can omit the variables $F(y, c)$ for all candidates $c \in \mathcal{A}$, because in these cases it also has to be $F(y, c) = 0$. Finally, the third requirement leads to the constraints $A_1 x = b_1$:

$$\sum_{c \in y \cap \mathcal{A}} F(y, c) = B(y) \quad \text{for each } y \in 2^{\mathcal{C}} \text{ such that } y \cap \mathcal{A} \neq \emptyset \quad (29)$$

$$\text{and } B(y) > 0$$

These constraints guarantee that the solution of the linear programming problem is a valid support distribution function.

An additional variable is needed, s , which is the support for the least supported candidate, and the goal of the linear programming problem is to maximize s .

Since s is the support for the least supported candidate, then for each candidate $c \in \mathcal{A}$ it holds that $Supp_F(c) - s \geq 0$, that is,

$$\sum_{y \in 2^{\mathcal{C}} : c \in y, B(y) > 0} F(y, c) - s \geq 0 \quad \text{for each } c \in \mathcal{A}. \quad (30)$$

This equation defines the constraints $A_2 x \geq b_2$ on the linear programming problem.

In summary, the following linear programming problem is equivalent to the problem of finding a support distribution function that maximizes the support for the least supported elected candidate for election σ and candidate subset \mathcal{A} .

$$\begin{aligned} & \text{maximize} && s \\ & \text{subject to} && \sum_{c \in y \cap \mathcal{A}} F(y, c) = B(y) && \text{for each } y \in 2^{\mathcal{C}} \text{ such that} \\ & && && y \cap \mathcal{A} \neq \emptyset \text{ and } B(y) > 0 && (31) \\ & && \sum_{y \in 2^{\mathcal{C}} : c \in y, B(y) > 0} F(y, c) - s \geq 0 && \text{for each } c \in \mathcal{A} \end{aligned}$$

For instance, for the sample election σ^a and candidate subset \mathcal{A}^a , the linear programming problem required to compute a support distribution function that maximizes the support for the least supported elected candidate would be:

$$\begin{aligned}
& \text{maximize} && s \\
& \text{subject to} && F(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) = 10,000 \\
& && F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) + F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) = 6,000 \\
& && F(\{\mathbf{c}\}, \mathbf{c}) = 5,500 \\
& && F(\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}, \mathbf{e}) = 5,000 \\
& && F(\{\mathbf{e}\}, \mathbf{e}) = 3,000 \\
& && F(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) + F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) - s \geq 0 \\
& && F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) + F(\{\mathbf{c}\}, \mathbf{c}) - s \geq 0 \\
& && F(\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}, \mathbf{e}) + F(\{\mathbf{e}\}, \mathbf{e}) - s \geq 0 \\
& && F(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}), F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}), F(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}), F(\{\mathbf{c}\}, \mathbf{c}) \geq 0 \\
& && F(\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}, \mathbf{e}), F(\{\mathbf{e}\}, \mathbf{e}) \geq 0
\end{aligned}$$

2. A well-known result in linear programming (e.g., see [7], page 19) is that each linear programming problem belongs to one and only one of the following cases.

- **Empty feasible region.** In this case, no combination of values exists for the variables that satisfy all of the constraints on the linear programming problem (including the nonnegativity constraints).
- **Unbounded optimal objective value.** In this case, the objective function can be made arbitrarily large (towards $+\infty$). Thus, no optimal solution exists.
- **One or several finite optimal solutions.** One or several points exist in \mathbb{R}^n that satisfy all of the constraints on the linear programming problem (including the nonnegativity constraints) and that maximize the objective function.

Next, we show that the linear programming problem defined by (31) is feasible and bounded, and thus at least one support distribution function exists that maximizes the support for the least supported candidate in \mathcal{A} .

To demonstrate that the linear programming problem defined in (31) is feasible, it is sufficient to find a (possibly nonoptimal) solution that satisfies all of the constraints. We assign $s = 0$. For each $y \in 2^{\mathcal{C}}$ such that $y \cap \mathcal{A} \neq \emptyset$ and $B(y) > 0$, pick one arbitrary candidate $c \in y \cap \mathcal{A}$. Then, $F(y, c) = B(y)$

and for all $c' \in (y \cap \mathcal{A}) - \{c\}$, define $F(y, c') = 0$. With these values, the variables satisfy all of the constraints on the linear programming problem. This includes the nonnegativity constraints because all of the variables take values greater than or equal to zero.

To show that the linear programming problem defined in (31) is bounded it is sufficient to note that s has to be less than or equal to $|\mathcal{V}|$. \square

For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, let $\mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$ be the (non-empty) set of support distribution functions that maximizes the support for the least supported candidate in \mathcal{A} for election σ :

$$\mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}} = \{F : F \in \mathfrak{F}_{\sigma, \mathcal{A}}, \forall c \in \mathcal{A}, \text{Supp}_F(c) \geq \max\text{Min}(\sigma, \mathcal{A})\}. \quad (32)$$

Now that we have introduced the concepts of $\max\text{Min}(\sigma, \mathcal{A})$ and $\mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$ and lemma 1, it is possible to present the first voting rule proposed in this study, which is defined in Algorithm 1.

Algorithm 1: Open D'Hondt (ODH) Method

Data: an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$

Result: the set of S elected candidates

```

1 begin
2    $\mathcal{C}_e = \emptyset$ ;
3   for  $i=1$  to  $S$  do
4     foreach  $c \in \mathcal{C} - \mathcal{C}_e$  do
5        $\mathcal{C}'_e = \mathcal{C}_e \cup \{c\}$ ;
6       choose any  $F \in \mathfrak{F}_{\sigma, \mathcal{C}'_e}^{\text{opt}}$ 
7        $s_c = \text{Supp}_F(c)$ ;
8     end
9     choose any  $w \in \mathcal{C} - \mathcal{C}_e$  such that  $s_w = \max_{c \in \mathcal{C} - \mathcal{C}_e} s_c$ ;
10     $\mathcal{C}_e = \mathcal{C}_e \cup \{w\}$ ;
11  end
12  return  $\mathcal{C}_e$ ;
13 end
```

The core of the ODH voting rule is found in lines 6, 7, and 9. Line 6 states that the total vote received by the candidates in \mathcal{C}'_e must be distributed in *any* way such that the support received by the least supported candidate in \mathcal{C}'_e is maximized. Then, the votes that have been distributed to the candidate under consideration are stored in s_c in line 7. Finally, in line 9, the candidate

w with the greatest value of s_w is selected.²

It should be noted that Algorithm 1 is not complete because it does not explain how to compute F . However, we can compute a solution for F using the linear programming problem defined in Lemma 1. This completes Algorithm 1.

It should also be noted that F may not be unique (for instance, for the sample election σ^a and candidate subset \mathcal{A}^a , there are infinitely many support distribution functions in $\mathfrak{F}_{\sigma^a, \mathcal{A}^a}^{\text{opt}}$).

Apparently, this fact might introduce a certain degree of uncertainty about the value of s_c , which could depend on the specific F that is selected. However, the results given in section 7 show that this is not the case because we prove that s_c must be equal to $\max\text{Min}(\sigma, \mathcal{C}'_e)$.

The following theorem establishes a bound for the complexity of computing ODH.

THEOREM 1 *ODH is computable in polynomial time. For any fully open election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, the number of arithmetic operations required to compute ODH is bounded by $O(|\mathcal{V}|^{4.5} S^{4.5} |C|(S + \log |\mathcal{V}|))$.*

Proof.

ODH requires that the loop in lines 3 to 11 in Algorithm 1 is executed S times, and that the inner loop in lines 4 to 8 is executed $|\mathcal{C} - \mathcal{C}_e|$ times. Thus, we need to show that the computation of F can be achieved in polynomial time.

In Lemma 1, we showed how F can be computed with a linear programming problem.

Any linear programming problem can be solved in polynomial time, which was proved by Khachian in [30]. Therefore, the linear programming problem defined to compute F can be solved in polynomial time.

A worst case complexity bound for solving the linear programming problem can be obtained using the algorithm proposed by Karmarkar for linear programming [29, 34, 22], where the computational complexity is lower than the complexity of the algorithm proposed by Khachian.

The number of arithmetic operations required to solve a linear programming problem with the algorithm proposed by Karmarkar is bounded by $O(n^{3.5}L)$, where n is the number of variables in the linear programming problem and L is the total number of bits required to represent the absolute values of all the nonzero coefficients that form the linear programming problem.

²As in the original D'Hondt method, when ties occur among two or more candidates, any of them can be selected. Like D'Hondt, ODH does not state how ties must be addressed.

The values of n and L can be bounded for the linear programming problem required to compute F as follows.

For n :

Each equation in A_1 requires that a certain number of votes must be distributed among certain candidates in \mathcal{C}'_e and a variable is defined for each of them. Since $|\mathcal{C}'_e| \leq S$, then each equation in A_1 requires at most S variables.

The total number of equations in A_1 cannot be greater than $|\mathcal{V}|$. It can be equal to $|\mathcal{V}|$ in the case where no two agents have cast identical votes. In summary, the total number of variables defined for A_1 cannot exceed $|\mathcal{V}|S$.

One more variable s is defined to represent the support for the least supported elected candidate.

It follows that $n \leq |\mathcal{V}|S + 1$.

For L :

- Each nonzero coefficient in A_1 is equal to 1, and therefore it can be represented with one bit. Therefore, the total number of bits required to represent the nonzero coefficients in A_1 cannot be greater than $|\mathcal{V}|S$.
- Each coefficient in b_1 is positive and smaller than or equal to $|\mathcal{V}|$, and thus it can be represented by at most $(1 + \log_2 |\mathcal{V}|)$ bits. Therefore, the total number of bits required to represent the coefficients in b_1 cannot be greater than $|\mathcal{V}|(1 + \log_2 |\mathcal{V}|)$ bits.
- Each constraint in A_2 is defined as requiring that the support for one candidate in \mathcal{C}'_e must be greater than or equal to s . One of these constraints exists for each candidate in \mathcal{C}'_e . Therefore, the total number of constraints in A_2 cannot be greater than S .

For each constraint in A_2 , the nonzero coefficients have a value of 1 for the variables corresponding to the shares of support received by the candidate in \mathcal{C}'_e and -1 for s . Therefore, the absolute value of each nonzero coefficient in A_2 can be represented by one bit. The total number of nonzero coefficients in each A_2 equation cannot be greater than $|\mathcal{V}|$ (to compute the support for the candidate under consideration) plus 1 (for s). Therefore, the total number of bits required to represent the nonzero coefficients in A_2 cannot be greater than $(|\mathcal{V}| + 1)S$.

- Each coefficient in b_2 is equal to 0. The total number of bits required to represent the nonzero coefficients in b_2 is 0.
- The total number of bits required to represent the nonzero coefficients in d^T is 1 (for s).

- Therefore, $L \leq |\mathcal{V}|S + |\mathcal{V}|(1 + \log_2 |\mathcal{V}|) + (|\mathcal{V}| + 1)S + 1 = O(|\mathcal{V}|(S + \log |\mathcal{V}|))$.

In conclusion, the number of arithmetic operations required to compute F in the worst case is bounded by $O(|\mathcal{V}|^{4.5}S^{3.5}(S + \log |\mathcal{V}|))$, and the number of arithmetic operations required to compute ODH in the worst case is bounded by $O(|\mathcal{V}|^{4.5}S^{4.5}|C|(S + \log |\mathcal{V}|))$. \square

As a final remark, it should be noted that the worst case complexity bound proved in Theorem 1 may well be outperformed in practical ODH implementations. It is well known that state-of-the-art linear programming solvers perform much better than the theoretical worst-case complexity in all but a few special cases, and thus it is probable that the performance of ODH in practice would be much better than the theoretical worst case bound proved in this study. An example of the size of linear programming problems that can be tackled in practice was demonstrated experimentally by Bixby *et al.* [9] in 1992, where a linear programming problem comprising 12,753,313 variables was solved in 4 minutes.

We conclude this section by illustrating ODH with an example.

6.1 Example

The execution of ODH over election σ^a requires the following steps.

1. $i = 1$. When the first candidate is selected, $\mathcal{C}_e = \emptyset$, and thus for any candidate $c \in \mathcal{C}$, $\mathcal{C}'_e = \{c\}$. There will be only one distribution support function $F \in \mathfrak{F}_{\sigma, \mathcal{C}'_e}$ that assigns candidate c with a support value that is equal to the sum of all the votes that she has received. In summary, the first selected candidate would have the most votes, which in the case of election σ^a is candidate **a**, who received a total of 16,000 votes.
2. $i = 2$. $\mathcal{C}_e = \{\mathbf{a}\}$. The support received by each of the remaining candidates is analyzed separately.

- Candidate **b**. $\mathcal{C}'_e = \{\mathbf{a}, \mathbf{b}\}$. The support for the least supported candidate is maximized when 4,000 votes from the agents that have selected $\{\mathbf{a}, \mathbf{b}\}$ are assigned to candidate **a** and 6,000 votes to candidate **b**. In this case, both candidates would receive 10,000 votes.

Intuitively, this assignment can be conceived as follows. Agents that select $\{\mathbf{a}, \mathbf{c}\}$ when faced with the choice between **a** and **b** would give all their votes to **a**. Similarly, agents that select $\{\mathbf{b}\}$

would give all their votes to **b**. Finally, agents that select $\{\mathbf{a}, \mathbf{b}\}$ would probably be happy to distribute their votes between **a** and **b** in such a manner that it would maximize the support received by both together, and thus the likelihood of both **a** and **b** being elected.

- Candidate **c**. $\mathcal{C}'_e = \{\mathbf{a}, \mathbf{c}\}$. The support for the least supported candidate is maximized when 750 votes from the agents that have selected $\{\mathbf{a}, \mathbf{c}\}$ are assigned to candidate **a** and 5,250 votes to candidate **c**. In this case, both candidates would receive 10,750 votes. Thus, it follows that candidate **c** would be elected ahead of candidate **b**.

It should be noted that if a voting rule M (different from ODH) selects a support distribution function $F \in \mathfrak{F}_{\sigma, \{\mathbf{a}, \mathbf{b}\}}$ such that the support for **b** would be greater than or equal to 10,750 votes, then this would imply that the support for **a** would be less than or equal to 9,250 votes. Therefore, when choosing **a** and **b**, the support for the least supported candidate **must be** less than the support received by both **a** and **c** given the vote distribution indicated in the previous paragraph.

Clearly, like D'Hondt, at each step, ODH chooses the candidate that maximizes the support obtained by the least supported elected candidate.

- Candidate **d**. $\mathcal{C}'_e = \{\mathbf{a}, \mathbf{d}\}$. Since **a** and **d** do not have any common agent, it is not possible that the support for **d** can exceed 9,500 votes, while **a** would retain the support of 16,000 votes. Again, it seems reasonable to consider that $\{\mathbf{a}, \mathbf{c}\}$ agents would be happy to distribute their votes such that both **a** and **c** beat **d**.
- The cases of **e**, **f**, and **g** are similar to the case of **d**, where they would receive 8,000, 5,000, and 5,000 votes, respectively.

Thus, the elected candidate would be **c**.

3. $i = 3$. $\mathcal{C}_e = \{\mathbf{a}, \mathbf{c}\}$. The support received by each of the remaining candidates is as follows.

- Candidate **b**. $\mathcal{C}'_e = \{\mathbf{a}, \mathbf{c}, \mathbf{b}\}$. The support for the least supported candidate is maximized when 5,500 votes from the agents that select $\{\mathbf{a}, \mathbf{b}\}$ are assigned to candidate **a** and 4,500 votes to candidate **b**, and when 3,000 votes from the agents that select $\{\mathbf{a}, \mathbf{c}\}$ are assigned to candidate **a** and 3,000 votes to candidate **c**. In this case, all three candidates would receive 8,500 votes.

- Candidate **d**. $\mathcal{C}'_e = \{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$. For $i = 2$, the maximum support for the least supported candidate is the total vote received by **d**, which is equal to 9,500 votes. Therefore, in this case, **d** (9,500 votes) beats **b** (8,500 votes).
- The cases of **e**, **f**, and **g** are also similar to the case of **d**, where they would receive 8,000, 5,000, and 5,000 votes, respectively.

The last elected candidate would then be **d**, and thus $\text{ODH}(\sigma^a) = \{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$.

7 ODH properties

In this section, we prove several theorems concerning ODH. The first states that s_c in Algorithm 1 is always equal to $\max\text{Min}(\sigma, \mathcal{C}'_e)$, regardless of the particular support distribution function selected. This is important for two reasons: 1) it guarantees that for a given election σ , several executions of ODH always produce the same result³; and 2) it guarantees that all of the candidates chosen after i iterations have at least the same support as the last candidate chosen, which forms the basis for proving that ODH satisfies the lower quota.

The remaining theorems state that ODH fulfills all the properties defined in section 4.3, i.e., it is equivalent to D'Hondt under closed lists, house monotonicity, lower quota satisfaction, and population monotonicity.

Before proving these theorems, it is necessary to prove a few intermediate lemmas.

LEMMA 2 *For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, and any candidate $c \in \mathcal{C} - \mathcal{A}$, it holds that:*

$$\max\text{Min}(\sigma, \mathcal{A} \cup \{c\}) \leq \max\text{Min}(\sigma, \mathcal{A}).$$

Proof.

Let F be a support distribution function in $\mathfrak{F}_{\sigma, (\mathcal{A} \cup \{c\})}^{\text{opt}}$. From F , it is possible to build a support distribution function $F_1 \in \mathfrak{F}_{\sigma, \mathcal{A}}$, such that for each candidate c_1 in \mathcal{A} it is $\text{Supp}_{F_1}(c_1) \geq \text{Supp}_F(c_1)$.

F assigns support to the candidates in \mathcal{A} plus to candidate c . We simply have to distribute in F_1 the support that F assigns to candidates in \mathcal{A} exactly in the same way as F . For the support that F assigns to candidate c there are

³Provided that the same candidate is selected in all executions when ties occur.

two possibilities: 1) if such support comes from agents that do not approve any candidate in \mathcal{A} , then such support cannot be distributed with F_1 ; 2) otherwise, we can distribute such support to the agents in \mathcal{A} in any way we like. One possible way of doing this is as follows.

$$\begin{aligned} F_1(y, c_1) &= 0 && \text{for each } y \in 2^{\mathcal{C}} \text{ and each } c_1 \in \mathcal{A} \\ &&& \text{such that } c_1 \notin y \text{ or } B(y) = 0 \\ F_1(y, c_1) &= F(y, c_1) + \frac{F(y, c)}{|y \cap \mathcal{A}|} && \text{for each } y \in 2^{\mathcal{C}} \text{ such that } B(y) > 0 \\ &&& \text{and } y \cap \mathcal{A} \neq \emptyset \text{ and each } c_1 \in y \cap \mathcal{A} \end{aligned} \quad (33)$$

It is not difficult to see that F_1 satisfies Equations (24) to (26) ($F_1(y, c_1) \geq 0$ for each (y, c_1) , $F_1(y, c_1) = 0$ for each (y, c_1) such that $c_1 \notin y$, and $\sum_{c_1 \in y \cap \mathcal{A}} F_1(y, c_1) = B(y)$ for each y such that $y \cap \mathcal{A} \neq \emptyset$). Since F_1 assigns each candidate in \mathcal{A} with all the votes assigned to the candidate by F (possibly including some votes assigned in F to c), then it follows that $\forall c_1 \in \mathcal{A}, \text{Supp}_{F_1}(c_1) \geq \text{Supp}_F(c_1)$, and thus:

$$\begin{aligned} \max\text{Min}(\sigma, \mathcal{A}) &\geq \min_{\forall c_1 \in \mathcal{A}} \text{Supp}_{F_1}(c_1) \geq \min_{\forall c_1 \in (\mathcal{A} \cup \{c\})} \text{Supp}_F(c_1) \\ &= \max\text{Min}(\sigma, (\mathcal{A} \cup \{c\})). \end{aligned} \quad (34)$$

□

In the following paragraphs we informally discuss the ideas on which the next lemmas are based.

Consider an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, a non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, and a support distribution function $F_1 \in \mathfrak{F}_{\sigma, \mathcal{A}}$. Suppose that the subset of the least supported candidates of \mathcal{A} under F_1 is composed of only one candidate ℓ . Suppose also that an agent a approves ℓ and another candidate $c_1 \in \mathcal{A}$, and that F_1 assigns the support of agent a to candidate c_1 . Clearly, it is possible to build another support distribution function in which (part of) the support of agent a would be assigned to candidate ℓ . We can do that in such a way that the support of candidate c_1 is still at least the same as the support of ℓ under the new support distribution function. Simply we have to assign to ℓ and c_1 a support of $k_1 = \min(1; (\text{Supp}_{F_1}(c_1) - \text{Supp}_{F_1}(\ell))/2) > 0$ and $1 - k_1$, respectively, coming from agent a . The other candidates receive the same support with the new support distribution function. ℓ may still be the least supported candidate or not but anyway the support of the least supported candidate under the new support distribution function is greater than the support of the least supported candidate under F_1 . Thus, $F_1 \notin \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$.

Consider now another support distribution function $F_2 \in \mathfrak{F}_{\sigma, \mathcal{A}}$, such that the subset of the least supported candidates of \mathcal{A} under F_2 is composed of

two candidates ℓ_1 and ℓ_2 . Suppose also that an agent a_1 approves both ℓ_1 and ℓ_2 , and that F_2 assigns the support of agent a_1 to candidate ℓ_2 . With this information there is not much we can do to increase the support of the least supported candidate, because if we assign part of the support from agent a_1 to ℓ_1 the result would be that the support of ℓ_2 decreases. Therefore, the support of the least supported candidate would decrease. But if another agent a_2 approves both ℓ_2 and a third candidate $c_2 \in \mathcal{A}$, and F_2 assigns the support of agent a_2 to c_2 we can increase the support of the least supported candidate. We can for instance assign to ℓ_2 and c_2 a support of $k_2 = \min(1; (Supp_{F_2}(c_2) - Supp_{F_2}(\ell_2))/2) > 0$ and $1 - k_2$, respectively, coming from agent a_2 and to assign to ℓ_1 and ℓ_2 a support of $k_2/2 > 0$ and $1 - k_2/2$, respectively, coming from agent a_1 . Again, the support of the least supported candidate would increase, and thus also it is $F_2 \notin \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$.

The discussion in the previous two paragraphs should give us the intuition that for each support distribution function F that belongs to $\mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$ a non-empty subset \mathcal{K} of the least supported candidates under F must exist such that they cannot receive additional support directly or indirectly that F assigns to other candidates. This means that F assigns the support of all the agents that approve some candidates in \mathcal{K} only between them. This will be addressed in Lemma 3. Later, in Corollary 1, we will show that \mathcal{K} does not depend on the particular F .

Finally, in Lemma 4 we will see that for each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, and each non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, the maximum support of the least supported candidate is determined by the subset \mathcal{K} in such a way that if we remove a candidate from \mathcal{A} that does not belong to \mathcal{K} the maximum support of the least supported candidate does not change. In particular, if for certain support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$ and certain candidate $c^* \in \mathcal{A}$ it is $Supp_F(c^*) > \max\text{Min}(\sigma, \mathcal{A})$ we can be sure that $c^* \notin \mathcal{K}$ because all the candidates in \mathcal{K} have to be in the set of the least supported candidates under F , and therefore it is $\max\text{Min}(\sigma, \mathcal{A} - \{c^*\}) = \max\text{Min}(\sigma, \mathcal{A})$.

Component of σ^b	Value
$ \mathcal{V} $	44,600
\mathcal{C}	{a, b, c, d, e, f, g}
S	3
B	See table 5

Table 4: Components of election σ^b

Tables 4 and 5 show the components of election σ^b , that will be used to

Candidates	Number of votes
{a, b}	10,000
{a, c}	6,000
{b}	4,000
{c}	5,500
{b, d, e}	600
{d}	9,500
{d, f, g}	9,000

Table 5: Result of election σ^b

illustrate Lemma 3.

We will consider the candidate subset $\mathcal{A}^b = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{f}\}$. We will also consider the support distribution functions $F^{b1}, F^{b2} \in \mathfrak{F}_{\sigma^b, \mathcal{A}^b}$, defined in Table 6. For all pairs $(y, c) \in 2^{\mathcal{C}} \times \mathcal{A}^b$ that are not shown in Table 6 it is $F^{b1}(y, c) = F^{b2}(y, c) = 0$.

$2^{\mathcal{C}}$	\mathcal{A}^b	F^{b1}	F^{b2}
{a, b}	a	5,900	5,500
{a, b}	b	4,100	4,500
{a, c}	a	2,800	3,000
{a, c}	c	3,200	3,000
{b}	b	4,000	4,000
{c}	c	5,500	5,500
{b, d, e}	b	600	0
{b, d, e}	d	0	600
{d}	d	9,500	9,500
{d, f, g}	d	300	500
{d, f, g}	f	8,700	8,500

Table 6: Support distribution functions F^{b1} and F^{b2}

The support values for F^{b1} are: $Supp_{F^{b1}}(\mathbf{a}) = 8,700$, $Supp_{F^{b1}}(\mathbf{b}) = 8,700$, $Supp_{F^{b1}}(\mathbf{c}) = 8,700$, $Supp_{F^{b1}}(\mathbf{d}) = 9,800$, and $Supp_{F^{b1}}(\mathbf{f}) = 8,700$.

The support values for F^{b2} are: $Supp_{F^{b2}}(\mathbf{a}) = 8,500$, $Supp_{F^{b2}}(\mathbf{b}) = 8,500$, $Supp_{F^{b2}}(\mathbf{c}) = 8,500$, $Supp_{F^{b2}}(\mathbf{d}) = 10,600$, and $Supp_{F^{b2}}(\mathbf{f}) = 8,500$.

Observe that F^{b1} belongs to $\mathfrak{F}_{\sigma^b, \mathcal{A}^b}^{\text{opt}}$, because it is not possible to increase simultaneously the support of \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{f} , above 8,700. However, F^{b2} does not belong to $\mathfrak{F}_{\sigma^b, \mathcal{A}^b}^{\text{opt}}$.

LEMMA 3 For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, and any support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$, a non-empty subset \mathcal{K} of \mathcal{A} exists such that

1. $\forall c \in \mathcal{K}, \text{Supp}_F(c) = \max \text{Min}(\sigma, \mathcal{A})$, and
2. $\forall c \in (\mathcal{A} - \mathcal{K}), \forall y \in 2^{\mathcal{C}}, y \cap \mathcal{K} \neq \emptyset \Rightarrow F(y, c) = 0$.

Proof.

Along the proof, we will use F^{b1} and F^{b2} to illustrate the differences between support distribution functions that belong to $\mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$ and those that do not belong to $\mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$.

We will also make use several times of the following idea. Consider an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, a non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, and a support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}$. As we know, for each $y \in 2^{\mathcal{C}}$ and each $c \in \mathcal{A}$, $F(y, c)$ is the fraction of all the agents that approve only all the candidates in y that F assigns to candidate c . Suppose that for certain candidate subset $y' \in 2^{\mathcal{C}}$ and for certain candidate $c' \in \mathcal{A}$ it is $F(y', c') = k > 0$. This means that 1) candidate c' has to belong to y' ; 2) at least $\lceil k \rceil$ agents approve all the candidates in y' ; and 3) F assigns k units of support corresponding to such agents to candidate c' . Suppose also that there exists other candidate $c'' \in \mathcal{A}$ who also belongs to y' . Therefore it is $\{c', c''\} \subseteq y'$. It is then possible to build another support distribution function $F' \in \mathfrak{F}_{\sigma, \mathcal{A}}$ to transfer k' units of support (with $0 < k' \leq k$) from candidate c' to candidate c'' . We simply have to make $F'(y', c'') = F(y', c'') + k'$, $F'(y', c') = F(y', c') - k'$, and $F'(y, c) = F(y, c)$, for each other (y, c) . Clearly, it is $\text{Supp}_{F'}(c'') = \text{Supp}_F(c'') + k'$, $\text{Supp}_{F'}(c') = \text{Supp}_F(c') - k'$, and $\text{Supp}_{F'}(c) = \text{Supp}_F(c)$ for each other candidate $c \in \mathcal{A}$.

Let \mathcal{L} be the set of the least supported candidates in \mathcal{A} under F .

$$\mathcal{L} = \{\ell : \ell \in \mathcal{A}, \text{Supp}_F(\ell) = \min_{c \in \mathcal{A}} \text{Supp}_F(c)\} \quad (35)$$

For F^{b1} and F^{b2} we have $\mathcal{L}(F^{b1}) = \mathcal{L}(F^{b2}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}\}$.

For each $\ell \in \mathcal{L}$, let $K(\ell)$ (kernel of ℓ) be the subset of \mathcal{L} computed as described by Algorithm 2.

For instance, for F^{b1} , the value of $K(\mathbf{c})$ is computed as follows:

1. Initially, $K(\mathbf{c}) = \{\mathbf{c}\}$.
2. At iteration 1, we add \mathbf{a} to $K(\mathbf{c})$, because $\mathbf{c} \in K(\mathbf{c})$, and $F^{b1}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) = 2,800 > 0$. Therefore, after iteration 1, $K(\mathbf{c}) = \{\mathbf{a}, \mathbf{c}\}$.

Algorithm 2: Computation of $K(\ell)$

Data: $(\ell, \mathcal{L}, \mathcal{C}, F)$

Result: $K(\ell)$ (kernel of ℓ)

begin

$K(\ell) = \{\ell\};$

repeat

 add to $K(\ell)$ all $\ell_1 \in (\mathcal{L} - K(\ell))$ such that

$\exists y \in 2^{\mathcal{C}}, \exists \ell_2 \in K(\ell), \{\ell_1, \ell_2\} \subseteq y, F(y, \ell_1) > 0 ;$

until *no element is added to $K(\ell)$ in one loop;*

return $K(\ell) ;$

end

3. At iteration 2, we add \mathbf{b} to $K(\mathbf{c})$, because $\mathbf{a} \in K(\mathbf{c})$, and $F^{b1}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b}) = 4, 100 > 0$. Therefore, after iteration 2, $K(\mathbf{c}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.
4. At iteration 3 it is not possible to add further elements to $K(\mathbf{c})$. Therefore, the final value of $K(\mathbf{c})$ for F^{b1} is $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

In fact, the values of K for F^{b1} and F^{b2} are the same and are: $K(\mathbf{a}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $K(\mathbf{b}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $K(\mathbf{c}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, and $K(\mathbf{f}) = \{\mathbf{f}\}$.

$K(\ell)$ is equal to ℓ plus the set of all other candidates in \mathcal{L} that can transfer positive shares of support to ℓ , either directly or in several hops. It follows that the candidates in $\mathcal{L} - K(\ell)$ receive zero support under F from agents who have chosen any candidate in $K(\ell)$ (otherwise they would be in $K(\ell)$), i.e.,

$$\forall \ell \in \mathcal{L}, \forall \ell' \in (\mathcal{L} - K(\ell)), \forall y \in 2^{\mathcal{C}}, y \cap K(\ell) \neq \emptyset \Rightarrow F(y, \ell') = 0. \quad (36)$$

Then, an $\ell_3 \in \mathcal{L}$ must exist such that:

$$\forall c \in (\mathcal{A} - \mathcal{L}), \forall y \in 2^{\mathcal{C}}, y \cap K(\ell_3) \neq \emptyset \Rightarrow F(y, c) = 0. \quad (37)$$

The existence of ℓ_3 is now proved by reduction to absurdity. If (37) does not hold for all $\ell \in \mathcal{L}$, then it would be possible to build a support distribution function F_1 such that the support for the least supported candidate under F_1 would be strictly greater than the support for the least supported candidate under F , as follows.

- For each candidate $\ell \in \mathcal{L}$, by hypothesis, a candidate $c_\ell \in \mathcal{A} - \mathcal{L}$ and a candidate subset $y_\ell \in 2^{\mathcal{C}}$ must exist such that $y_\ell \cap K(\ell) \neq \emptyset$ and $F(y_\ell, c_\ell) > 0$. Since $F(y_\ell, c_\ell) > 0$ it has to be $c_\ell \subseteq y_\ell$ (an agent cannot give support to a candidate that she does not approve).

- Since $y_\ell \cap K(\ell) \neq \emptyset$ there are two possibilities: either $\ell \in y_\ell$ or there exists a $\ell^{(n)} \in K(\ell)$, $\ell^{(n)} \neq \ell$ such that $\ell^{(n)} \in y_\ell$. In the second case, since $\ell^{(n)}$ belongs to $K(\ell)$, a sequence $\ell^{(1)}, \dots, \ell^{(n)}$ (with $\ell = \ell^{(1)}$) and a sequence $y_\ell^1, \dots, y_\ell^{(n-1)}$ must exist such that:

$$\begin{aligned} \forall i = 1, \dots, n, \ell^{(i)} &\in K(\ell) \\ \forall i = 1, \dots, (n-1), \{\ell^{(i)}, \ell^{(i+1)}\} &\subseteq y_\ell^i \\ \forall i = 1, \dots, (n-1), F(y_\ell^i, \ell^{(i+1)}) &> 0. \end{aligned} \tag{38}$$

- Let k , $k_1(\ell)$, and $k_2(\ell)$ be

$$\begin{aligned} k &= \left(\min_{\forall c \in (\mathcal{A} - \mathcal{L})} \text{Supp}_F(c) \right) - \min_{\forall c \in \mathcal{A}} \text{Supp}_F(c) > 0 \\ k_1(\ell) &= \begin{cases} \min\{F(y_\ell^i, \ell^{(i+1)}), i = 1, \dots, (n-1)\} & \text{if } \ell \notin y_\ell \\ \text{otherwise } k_1(\ell) \text{ is not defined} \end{cases} \\ k_2(\ell) &= \begin{cases} \min\left\{\frac{F(y_\ell, c_\ell)}{|\mathcal{L}|}; \frac{k_1(\ell)}{|\mathcal{L}|}; \frac{k}{3|\mathcal{L}|}\right\} & \text{if } \ell \notin y_\ell \\ \min\left\{\frac{F(y_\ell, c_\ell)}{|\mathcal{L}|}; \frac{k}{3|\mathcal{L}|}\right\} & \text{if } \ell \in y_\ell \end{cases} \end{aligned} \tag{39}$$

- There are two cases now:
 - When $\ell \notin y_\ell$, it is possible to transfer $k_2(\ell)$ units of support from c_ℓ to $\ell^{(n)}$, and then to ℓ through $\ell^{(n-1)}, \dots, \ell^{(2)}$. To achieve this, $F(y_\ell, c_\ell)$ must be decreased in $k_2(\ell)$ and $F(y_\ell, \ell^{(n)})$ must also be increased in $k_2(\ell)$. In addition, for each y_ℓ^i , decrease $F(y_\ell^i, \ell^{(i+1)})$ and increase $F(y_\ell^i, \ell^{(i)})$ in $k_2(\ell)$.
Observe that the support received by each of $\ell^{(n)}, \dots, \ell^{(2)}$ would not change with these operations and that the support of c_ℓ decreases in $k_2(\ell)$ units and the support of $\ell = \ell^{(1)}$ increases in $k_2(\ell)$ units.
 - When $\ell \in y_\ell$, it is possible to transfer $k_2(\ell)$ units of support from c_ℓ directly to ℓ . Simply decrease $F(y_\ell, c_\ell)$ in $k_2(\ell)$ and increase $F(y_\ell, \ell)$ also in $k_2(\ell)$.
- The way that we choose $k_2(\ell)$ guarantees that the support for c_ℓ under F_1 does not fall below $\text{Supp}_F(c_\ell) - k/3$ (even if the same c_ℓ is used for several candidates in \mathcal{L}). It also guarantees that the support for all candidates $\ell \in \mathcal{L}$ under F_1 is strictly greater than their support under F , but not greater than $\text{Supp}_F(\ell) + k/(3|\mathcal{L}|)$.

Therefore, the support for all candidates in $\mathcal{A} - \mathcal{L}$ under F_1 will be greater than the support for all candidates in \mathcal{L} under F_1 :

$$\forall c_1 \in \mathcal{A} - \mathcal{L}, \forall c_2 \in \mathcal{L}, \text{Supp}_{F_1}(c_1) - \text{Supp}_{F_1}(c_2) \geq (\text{Supp}_F(c_1) - \frac{k}{3|\mathcal{L}|}|\mathcal{L}|) - (\text{Supp}_F(c_2) + \frac{k}{3|\mathcal{L}|}) = (\text{Supp}_F(c_1) - \text{Supp}_F(c_2)) - \frac{k}{3} - \frac{k}{3|\mathcal{L}|} \geq k - \frac{k}{3} - \frac{k}{3|\mathcal{L}|} \geq \frac{k}{3}.$$

Moreover, $k_2(\ell)$ also guarantees that for all $(y_\ell^i, \ell^{(i+1)})$, it holds that $F_1(y_\ell^i, \ell^{(i+1)}) \geq 0$ (even if the same pair $(y_\ell^i, \ell^{(i+1)})$ is used in the transfer of support to several candidates in \mathcal{L}), and that $F_1(y_\ell, c_\ell) \geq 0$ (even if the same pair (y_ℓ, c_ℓ) is also used in several transfers of support), as required for any support distribution function.

F_1 contradicts the hypothesis that $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$, and thus the existence of a candidate $\ell_3 \in \mathcal{L}$ that fulfills (37) is proved.

We illustrate these ideas with F^{b2} . We are going to see that because for F^{b2} equation (37) does not hold for all $\ell \in \mathcal{L}(F^{b2})$, it is possible to build a support distribution function F^{b3} following the strategy we have just described, such that the support for the least supported candidate under F^{b3} is strictly greater than the support for the least supported candidate under F^{b2} .

First of all, observe that, indeed, equation (37) does not hold for all $\ell \in \mathcal{L}(F^{b2})$. We have

- For **a**, it is **d** $\in (\mathcal{A}^b - \mathcal{L}(F^{b2}))$, **b** $\in K(\mathbf{a})$, and $F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d}) = 600 > 0$,
- For **b**, it is **d** $\in (\mathcal{A}^b - \mathcal{L}(F^{b2}))$, **b** $\in K(\mathbf{b})$, and $F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d}) = 600 > 0$,
- For **c**, it is **d** $\in (\mathcal{A}^b - \mathcal{L}(F^{b2}))$, **b** $\in K(\mathbf{c})$, and $F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d}) = 600 > 0$, and
- For **f**, it is **d** $\in (\mathcal{A}^b - \mathcal{L}(F^{b2}))$, **f** $\in K(\mathbf{f})$, and $F^{b2}(\{\mathbf{d}, \mathbf{f}, \mathbf{g}\}, \mathbf{d}) = 500 > 0$.

We also have:

- $k = \left(\min_{\forall c \in (\mathcal{A}^b - \mathcal{L}(F^{b2}))} \text{Supp}_{F^{b2}}(c) \right) - \min_{\forall c \in \mathcal{A}^b} \text{Supp}_{F^{b2}}(c) = 10,600 - 8,500 = 2,100$.
- $|\mathcal{L}(F^{b2})| = |\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}\}| = 4$.

- For **a**: $\ell^{(2)} = \mathbf{b}$, $\ell^{(1)} = \mathbf{a}$, $y_{\mathbf{a}}^1 = \{\mathbf{a}, \mathbf{b}\}$, $F^{b2}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b}) = 4,500 > 0$.
 $k_1(\mathbf{a}) = \min(F^{b2}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b})) = \min(4,500) = 4,500$.
 $k_2(\mathbf{a}) = \min(\frac{F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d})}{|\mathcal{L}(F^{b2})|}; \frac{k_1(\mathbf{a})}{|\mathcal{L}(F^{b2})|}; \frac{k}{3|\mathcal{L}(F^{b2})|}) = \min(150; 1,125; 175) = 150$.
- For **b**, k_1 is not defined because in this case it is possible to transfer support directly from **d** to **b**, without intermediate steps.
 $k_2(\mathbf{b}) = \min(\frac{F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d})}{|\mathcal{L}(F^{b2})|}; \frac{k}{3|\mathcal{L}(F^{b2})|}) = \min(150; 175) = 150$.
- For **c**, $\ell^{(3)} = \mathbf{b}$, $\ell^{(2)} = \mathbf{a}$, $\ell^{(1)} = \mathbf{c}$, $y_{\mathbf{c}}^2 = \{\mathbf{a}, \mathbf{b}\}$, $y_{\mathbf{c}}^1 = \{\mathbf{a}, \mathbf{c}\}$, $F^{b2}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b}) = 4,500 > 0$, $F^{b2}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) = 3,000 > 0$.
 $k_1(\mathbf{c}) = \min(F^{b2}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b}); F^{b2}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a})) = \min(4,500; 3,000) = 3,000$.
 $k_2(\mathbf{c}) = \min(\frac{F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d})}{|\mathcal{L}(F^{b2})|}; \frac{k_1(\mathbf{c})}{|\mathcal{L}(F^{b2})|}; \frac{k}{3|\mathcal{L}(F^{b2})|}) = \min(150; 750; 175) = 150$.
- For **f**, k_1 is not defined because in this case it is possible to transfer support directly from **d** to **f**, without intermediate steps.
 $k_2(\mathbf{f}) = \min(\frac{F^{b2}(\{\mathbf{d}, \mathbf{f}, \mathbf{g}\}, \mathbf{d})}{|\mathcal{L}(F^{b2})|}; \frac{k}{3|\mathcal{L}(F^{b2})|}) = \min(125; 175) = 125$.

With all this data it is possible to define now F^{b3} :

- $F^{b3}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) = F^{b2}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) + k_2(\mathbf{a}) + k_2(\mathbf{c}) = 5,500 + 150 + 150 = 5,800$.
- $F^{b3}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b}) = F^{b2}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{b}) - k_2(\mathbf{a}) - k_2(\mathbf{c}) = 4,500 - 150 - 150 = 4,200$.
- $F^{b3}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) = F^{b2}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) - k_2(\mathbf{c}) = 3,000 - 150 = 2,850$.
- $F^{b3}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) = F^{b2}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) + k_2(\mathbf{c}) = 3,000 + 150 = 3,150$.
- $F^{b3}(\{\mathbf{b}\}, \mathbf{b}) = F^{b2}(\{\mathbf{b}\}, \mathbf{b}) = 4,000$.
- $F^{b3}(\{\mathbf{c}\}, \mathbf{c}) = F^{b2}(\{\mathbf{c}\}, \mathbf{c}) = 5,500$.
- $F^{b3}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{b}) = F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{b}) + k_2(\mathbf{a}) + k_2(\mathbf{b}) + k_2(\mathbf{c}) = 0 + 150 + 150 + 150 = 450$.
- $F^{b3}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d}) = F^{b2}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d}) - k_2(\mathbf{a}) - k_2(\mathbf{b}) - k_2(\mathbf{c}) = 600 - 150 - 150 - 150 = 150$.
- $F^{b3}(\{\mathbf{d}\}, \mathbf{d}) = F^{b2}(\{\mathbf{d}\}, \mathbf{d}) = 9,500$.

- $F^{b3}(\{\mathbf{d}, \mathbf{f}, \mathbf{g}\}, \mathbf{d}) = F^{b2}(\{\mathbf{d}, \mathbf{f}, \mathbf{g}\}, \mathbf{d}) - k_2(\mathbf{f}) = 500 - 125 = 375.$
- $F^{b3}(\{\mathbf{d}, \mathbf{f}, \mathbf{g}\}, \mathbf{f}) = F^{b2}(\{\mathbf{d}, \mathbf{f}, \mathbf{g}\}, \mathbf{f}) + k_2(\mathbf{f}) = 8,500 + 125 = 8,625.$

The support values for F^{b3} are:

- $Supp_{F^{b3}}(\mathbf{a}) = 8,650.$
- $Supp_{F^{b3}}(\mathbf{b}) = 8,650.$
- $Supp_{F^{b3}}(\mathbf{c}) = 8,650.$
- $Supp_{F^{b3}}(\mathbf{d}) = 10,025.$
- $Supp_{F^{b3}}(\mathbf{f}) = 8,625.$

As expected, the support for the least supported candidate under F^{b3} (8,625 for candidate \mathbf{f}) is strictly greater than the support for the least supported candidate under F^{b2} (8,500 for candidates \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{f}). As it can be seen, what we have done is to transfer support from candidate \mathbf{d} directly or in several steps to candidates \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{f} .

Observe also that the support of candidate \mathbf{d} under F^{b3} (10,025) is still strictly greater than the support under F^{b3} of candidates \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{f} .

Observe now that in the case of F^{b1} it is not possible to follow a similar strategy to increase the support of the least supported candidate because for all candidates $c \in \mathcal{A}^b - \mathcal{L}(F^{b1}) = \mathcal{A}^b - \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}\} = \{\mathbf{d}\}$, for all $y \in 2^{\mathcal{C}}$, $y \cap K(\mathbf{a}) = y \cap \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \neq \emptyset \Rightarrow F^{b1}(y, c) = 0$ (in particular, $F^{b1}(\{\mathbf{b}, \mathbf{d}, \mathbf{e}\}, \mathbf{d}) = 0$).

Continuing with the formal proof, we have seen (Equation (37)) that if $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$, then a candidate $\ell_3 \in \mathcal{L}$ must exist such that

$$\forall c \in (\mathcal{A} - \mathcal{L}), \forall y \in 2^{\mathcal{C}}, y \cap K(\ell_3) \neq \emptyset \Rightarrow F(y, c) = 0. \quad (40)$$

Moreover, Equation (36)) implies that

$$\forall c \in (\mathcal{L} - K(\ell_3)), \forall y \in 2^{\mathcal{C}}, y \cap K(\ell_3) \neq \emptyset \Rightarrow F(y, c) = 0, \quad (41)$$

and thus,

$$\forall c \in (\mathcal{A} - K(\ell_3)), \forall y \in 2^{\mathcal{C}}, y \cap K(\ell_3) \neq \emptyset \Rightarrow F(y, c) = 0. \quad (42)$$

We can define $\mathcal{K} = K(\ell_3)$. It can be easily seen that \mathcal{K} satisfies the conditions of the lemma. The first condition is that the support under F of all candidates in \mathcal{K} is $\max\text{Min}(\sigma, \mathcal{A})$. This is guaranteed by the definition of algorithm 2. The second condition is precisely Equation (42). \square

The following interesting corollary can also be obtained from the proof of Lemma 3.

COROLLARY 1 *For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, a non-empty subset \mathcal{K} of \mathcal{A} exists such that:*

$$1. \forall F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}, \forall c \in \mathcal{K}, \text{Supp}_F(c) = \max\text{Min}(\sigma, \mathcal{A})$$

$$2. \max\text{Min}(\sigma, \mathcal{A}) = \frac{\sum_{y \in 2^{\mathcal{C}}: y \cap \mathcal{K} \neq \emptyset} B(y)}{|\mathcal{K}|}.$$

Proof.

Pick any support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$ and let \mathcal{K} be the candidate subset whose existence is proved in Lemma 3 for each $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$.

In Lemma 3, we showed that: (i) the support distribution function F distributes all the votes from the agents selecting any candidate in \mathcal{K} only between them; and (ii) that the support received for each of the candidates in \mathcal{K} under the support distribution function considered in Lemma 3 is equal to $\max\text{Min}(\sigma, \mathcal{A})$. This means that the total number of agents who approve at least one of the candidates in \mathcal{K} must be equal to $|\mathcal{K}| \max\text{Min}(\sigma, \mathcal{A})$. This proves the second part of the corollary.

For the first part of the corollary, consider any other support distribution function $F' \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$. If for any candidate $c \in \mathcal{K}$, this is $\text{Supp}_{F'}(c) > \max\text{Min}(\sigma, \mathcal{A})$, then for some other candidate $c' \in \mathcal{K}$, it would be $\text{Supp}_{F'}(c') < \max\text{Min}(\sigma, \mathcal{A})$ (otherwise the total number of votes for any of the candidates in \mathcal{K} cannot be equal to $\max\text{Min}(\sigma, \mathcal{A})|\mathcal{K}|$). This contradicts the hypothesis that $F' \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$, which proves the first part of the corollary. \square

LEMMA 4 *For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, any non-empty candidate subset $\mathcal{A} \subseteq \mathcal{C}$, and any support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{A}}^{\text{opt}}$, if any $c^* \in \mathcal{A}$ exists such that $\text{Supp}_F(c^*) > \max\text{Min}(\sigma, \mathcal{A})$, then*

$$\max\text{Min}(\sigma, \mathcal{A}) = \max\text{Min}(\sigma, \mathcal{A} - \{c^*\}).$$

Proof.

1. Consider the non-empty candidate subset \mathcal{K} whose existence was proved in Lemma 3, and for which it holds that

$$Supp_F(c) = \max\text{Min}(\sigma, \mathcal{A}), \text{ for each } c \in \mathcal{K}, \text{ and} \quad (43)$$

$$F(y, c) = 0 \quad \text{for each } c \in (\mathcal{A} - \mathcal{K}), \quad (44)$$

and for each $y \in 2^{\mathcal{C}}$ such that $y \cap \mathcal{K} \neq \emptyset$.

All of the candidates in $\mathcal{A} - \mathcal{K}$ receive zero support under F from agents who select any candidate in \mathcal{K} , and thus F distributes all the votes from agents who have selected any candidate in \mathcal{K} only between them (even if these agents have also selected other candidates that are not in \mathcal{K}). Therefore, it is possible to define a support distribution function $F_1 \in \mathfrak{F}_{\sigma, \mathcal{K}}$ to distribute votes between the candidates in \mathcal{K} in the same way as F :

$$F_1(y, c) = F(y, c) \text{ for each } y \in 2^{\mathcal{C}}, \text{ for each } c \in \mathcal{K}. \quad (45)$$

F_1 satisfies the requirements that a support distribution function must fulfill, defined in Equations (24) to (26) in section 5 ($F_1(y, c) \geq 0$ for each (y, c) , $F_1(y, c) = 0$ for each (y, c) such that $c \notin y$, and $\sum_{c \in y \cap \mathcal{K}} F_1(y, c) = B(y)$ for each y such that $y \cap \mathcal{K} \neq \emptyset$). In particular, for the third requirement, and taking into account Equation (44), we have

$$\begin{aligned} \sum_{c \in y \cap \mathcal{K}} F_1(y, c) &= \sum_{c \in y \cap \mathcal{K}} F(y, c) = \sum_{c \in y \cap \mathcal{K}} F(y, c) + 0 = \\ &= \sum_{c \in y \cap \mathcal{K}} F(y, c) + \sum_{c \in y \cap (\mathcal{A} - \mathcal{K})} F(y, c) \\ &= \sum_{c \in y \cap \mathcal{A}} F(y, c) = B(y) \end{aligned} \quad (46)$$

for each y such that $y \cap \mathcal{K} \neq \emptyset$

The last equality in Equation (46) comes from the fact that F is a support distribution function of election σ and candidate subset \mathcal{A} , and therefore it must also satisfy the corresponding third requirement ($\sum_{c \in y \cap \mathcal{A}} F(y, c) = B(y)$ for each y such that $y \cap \mathcal{A} \neq \emptyset$).

Clearly, the candidates in \mathcal{K} receive the same support under F_1 as that under F :

$$\forall c \in \mathcal{K}, Supp_{F_1}(c) = Supp_F(c). \quad (47)$$

Also, as stated at the beginning of the lemma, the candidates in \mathcal{K} verify that

$$\forall c \in \mathcal{K}, Supp_F(c) = \max\text{Min}(\sigma, \mathcal{A}). \quad (48)$$

As all of the candidates in \mathcal{K} receive the same support under F_1 , it must be $F_1 \in \mathfrak{F}_{\sigma, \mathcal{K}}^{\text{opt}}$, and thus

$$\forall c \in \mathcal{K}, \text{Supp}_{F_1}(c) = \max\text{Min}(\sigma, \mathcal{K}) \quad (49)$$

Finally, combining Equations (47), (48) and (49) we have:

$$\max\text{Min}(\sigma, \mathcal{K}) = \max\text{Min}(\sigma, \mathcal{A}) \quad (50)$$

2. Clearly $c^* \notin \mathcal{K}$, because for all candidates $c \in \mathcal{K}$ it is $\text{Supp}_F(c) = \max\text{Min}(\sigma, \mathcal{A})$ and the lemma establishes that $\text{Supp}_F(c^*) > \max\text{Min}(\sigma, \mathcal{A})$. Therefore, $c^* \in \mathcal{A} - \mathcal{K}$.

Consider any ordering c_i of the candidates in $\mathcal{A} - \mathcal{K}$, $i = 1, \dots, m$, with $m = |\mathcal{A} - \mathcal{K}|$ such that $c_m = c^*$. Consider the sequence of candidate subsets defined as follows:

$$\mathcal{A}_0 = \mathcal{K} \quad (51)$$

$$\mathcal{A}_{(i+1)} = \mathcal{A}_i \cup \{c_{(i+1)}\}, i = 0, \dots, (m-1). \quad (52)$$

Note that $\mathcal{A}_{(m-1)} = \mathcal{A} - \{c^*\}$ and that $\mathcal{A}_m = \mathcal{A}$. Then, it follows from Lemma 2 that

$$\begin{aligned} \max\text{Min}(\sigma, \mathcal{A}_0) &\geq \max\text{Min}(\sigma, \mathcal{A}_1) \geq \dots \geq \max\text{Min}(\sigma, \mathcal{A}_{(m-1)}) \\ &\geq \max\text{Min}(\sigma, \mathcal{A}_m). \end{aligned} \quad (53)$$

From Equation (50) we know that $\max\text{Min}(\sigma, \mathcal{K}) = \max\text{Min}(\sigma, \mathcal{A})$. But as $\mathcal{A}_0 = \mathcal{K}$ and $\mathcal{A}_m = \mathcal{A}$, it is $\max\text{Min}(\sigma, \mathcal{A}_0) = \max\text{Min}(\sigma, \mathcal{A}_m)$. Combining this with Equation (53) we have

$$\begin{aligned} \max\text{Min}(\sigma, \mathcal{A}_0) &= \max\text{Min}(\sigma, \mathcal{A}_1) = \dots = \max\text{Min}(\sigma, \mathcal{A}_{(m-1)}) \\ &= \max\text{Min}(\sigma, \mathcal{A}_m), \end{aligned} \quad (54)$$

and thus

$$\max\text{Min}(\sigma, \mathcal{A}) = \max\text{Min}(\sigma, \mathcal{A} - \{c^*\}). \quad (55)$$

□

It is now possible to prove the following theorem. This theorem shows that ODH does not depend on the particular support distribution function F chosen. It will also be used later to prove that ODH satisfies the lower quota.

THEOREM 2 For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any $i = 0, \dots, (S - 1)$, let \mathcal{C}_e^i be the set of the first i candidates chosen by ODH (when $i = 0$, $\mathcal{C}_e^0 = \emptyset$).

Then, $\forall c \in \mathcal{C} - \mathcal{C}_e^i, \forall F \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^i \cup \{c\})}^{\text{opt}}$, it holds that $\text{Supp}_F(c) = \max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\})$.

Proof.

The proof is obtained by induction.

Base case: $i = 0$. Clearly, when $i = 0$ the theorem holds because $\mathcal{C}_e^0 \cup \{c\} = \{c\}$, and for the unique $F \in \mathfrak{F}_{\sigma, \{c\}}^{\text{opt}}$ it must be true that $\text{Supp}_F(c) = \max\text{Min}(\sigma, \{c\})$.

Inductive step: Assuming that the theorem holds for $i = n$, then it must be proved that it also holds for $i = n + 1$.

This is achieved by reduction to absurdity. Suppose that for a certain $c \in \mathcal{C} - \mathcal{C}_e^{n+1}$, $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^{n+1} \cup \{c\})}^{\text{opt}}$ exists such that

$$\text{Supp}_F(c) > \max\text{Min}(\sigma, \mathcal{C}_e^{n+1} \cup \{c\}). \quad (56)$$

By Lemma 4, this means that

$$\max\text{Min}(\sigma, \mathcal{C}_e^{n+1} \cup \{c\}) = \max\text{Min}(\sigma, \mathcal{C}_e^{n+1}). \quad (57)$$

Let c_{n+1} be the $(n + 1)$ th candidate chosen by ODH for election σ . Thus, $\mathcal{C}_e^{n+1} = \mathcal{C}_e^n \cup \{c_{n+1}\}$. From F , it is possible to build a support distribution function $F_1 \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^n \cup \{c\})}$, such that for each candidate c' in $\mathcal{C}_e^n \cup \{c\}$ it is $\text{Supp}_{F_1}(c') \geq \text{Supp}_F(c')$.

F assigns support to the candidates in $\mathcal{C}_e^n \cup \{c\}$ plus to candidate c_{n+1} . We simply have to distribute in F_1 the support that F assigns to candidates in $\mathcal{C}_e^n \cup \{c\}$ exactly in the same way as F . For the support that F assigns to candidate c_{n+1} there are two possibilities: 1) if such support comes from agents that do not approve any candidate in $\mathcal{C}_e^n \cup \{c\}$, then such support cannot be distributed with F_1 ; 2) otherwise, we can distribute such support to the agents in $\mathcal{C}_e^n \cup \{c\}$ in any way we like.

One possible way to define F_1 is as follows.

$$\begin{aligned} F_1(y, c') &= 0 && \text{for each } y \in 2^{\mathcal{C}} \text{ and each } c' \in \mathcal{C}_e^n \cup \{c\} \\ &&& \text{such that } c' \notin y \text{ or } B(y) = 0 \\ F_1(y, c') &= \frac{F(y, c_{n+1})}{|y \cap (\mathcal{C}_e^n \cup \{c\})|} && \text{for each } y \in 2^{\mathcal{C}} \text{ such that } B(y) > 0 \\ &&& \text{and } y \cap (\mathcal{C}_e^n \cup \{c\}) \neq \emptyset \\ &&& \text{and each } c' \in y \cap (\mathcal{C}_e^n \cup \{c\}) \end{aligned} \quad (58)$$

F_1 satisfies Equations (24) to (26) ($F_1(y, c') \geq 0$ for each (y, c') , $F_1(y, c') = 0$ for each (y, c') such that $c' \notin y$, and $\sum_{c' \in y \cap (\mathcal{C}_e^n \cup \{c\})} F_1(y, c') = B(y)$ for each

y such that $y \cap (\mathcal{C}_e^n \cup \{c\}) \neq \emptyset$. Since F_1 assigns each candidate in $\mathcal{C}_e^n \cup \{c\}$ with all the votes that were assigned to the candidate by F (possibly including some votes assigned in F to c_{n+1}), then it must hold that

$$\forall c' \in \mathcal{C}_e^n \cup \{c\}, \text{Supp}_{F_1}(c') \geq \text{Supp}_F(c'). \quad (59)$$

Two possibilities are now considered. Either F_1 maximizes the support for the least supported candidate in $\mathcal{C}_e^n \cup \{c\}$ or it does not.

In the first case, we have

$$\max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c\}) = \min_{\forall c' \in (\mathcal{C}_e^n \cup \{c\})} \text{Supp}_{F_1}(c'), \quad (60)$$

and thus, by the inductive hypothesis,

$$\max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c\}) = \text{Supp}_{F_1}(c). \quad (61)$$

By combining (56), (57), (59), and (61), we have:

$$\begin{aligned} \max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c\}) &= \text{Supp}_{F_1}(c) \geq \text{Supp}_F(c) > \max\text{Min}(\sigma, \mathcal{C}_e^{n+1} \cup \{c\}) \\ &= \max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c_{n+1}\}). \end{aligned} \quad (62)$$

However, this is not possible because candidate c would have been selected by the ODH instead of c_{n+1} at iteration $(n+1)$.

Conversely, suppose that

$$\max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c\}) > \min_{\forall c' \in (\mathcal{C}_e^n \cup \{c\})} \text{Supp}_{F_1}(c'). \quad (63)$$

By combining (57), (59), and (63), and taking into account that since F belongs to $\mathfrak{F}_{\sigma, (\mathcal{C}_e^{n+1} \cup \{c\})}^{\text{opt}}$, it is $\min_{\forall c' \in (\mathcal{C}_e^{n+1} \cup \{c\})} \text{Supp}_F(c') = \max\text{Min}(\sigma, \mathcal{C}_e^{n+1} \cup \{c\})$, we have:

$$\begin{aligned} \max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c\}) &> \min_{\forall c' \in (\mathcal{C}_e^n \cup \{c\})} \text{Supp}_{F_1}(c') \geq \min_{\forall c' \in (\mathcal{C}_e^{n+1} \cup \{c\})} \text{Supp}_F(c') \\ &= \max\text{Min}(\sigma, \mathcal{C}_e^{n+1} \cup \{c\}) = \max\text{Min}(\sigma, \mathcal{C}_e^n \cup \{c_{n+1}\}). \end{aligned} \quad (64)$$

Again, this is not possible because candidate c would have been selected by the ODH instead of c_{n+1} at iteration $(n+1)$. \square

We can now present the proofs related to the social choice properties that ODH possesses. Henceforth, for any approval-based multi-winner election σ , let $ODH(\sigma)$ be the set of winners that ODH outputs for such election.

THEOREM 3 *The ODH method is equivalent to D'Hondt under closed lists. That is, for any election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ such that a set of candidate subsets $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ exists, for which it holds that:*

$$|\mathcal{V}| > 0 \quad (65)$$

$$\forall i = 1, \dots, n; \mathcal{C}_i \subseteq \mathcal{C} \quad (66)$$

$$\forall i = 1, \dots, n; |\mathcal{C}_i| \geq S \quad (67)$$

$$\forall i, j = 1, \dots, n; i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset \quad (68)$$

$$\forall \mathcal{C}' \in 2^{\mathcal{C}}, \mathcal{C}' \notin \{\mathcal{C}_1, \dots, \mathcal{C}_n\} \Rightarrow B(\mathcal{C}') = 0, \quad (69)$$

consider the closed list election $\sigma_1 = \langle \mathcal{V}, \mathcal{C}_c, S, \mathcal{L}, B_1 \rangle$, where:

$$\mathcal{L} = \{\mathcal{C}_1, \dots, \mathcal{C}_n\} \quad (70)$$

$$\mathcal{C}_c = \bigcup_{i=1}^n \mathcal{C}_i \quad (71)$$

$$\forall i = 1, \dots, n; B_1(\mathcal{C}_i) = B(\mathcal{C}_i). \quad (72)$$

In addition, for each 'list' \mathcal{C}_i , a specific ordering of the candidates must be provided. Then, $ODH(\sigma)$ can be equal to $DH(\sigma_1)$ (subject to how the ties between candidates are broken).

Proof.

Consider an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and a set of candidate subsets $\{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ that fulfills Equations (65) to (69). Candidates that are not in \mathcal{C}_c (defined as (71)) cannot be elected because they have not been selected by any agent.

As in the previous theorem, let \mathcal{C}_e^i be the set of the first i candidates chosen by ODH. Let c be a candidate in $\mathcal{C} - \mathcal{C}_e^i$ and let \mathcal{C}_j be the "list" to which c belongs. In Theorem 2 we proved that c will be always in the set of the least supported candidates when we maximize the support for the least supported candidate in $\mathcal{C}_e^i \cup \{c\}$. But c is approved only by the agents that approve all the candidates in \mathcal{C}_j , and no candidate in \mathcal{C}_j is approved by any other agent. This means that maximizing the support of the least supported candidate in $\mathcal{C}_e^i \cup \{c\}$ has to depend only of the number of agents that approve \mathcal{C}_j and of the number of candidates of \mathcal{C}_j that are in $\mathcal{C}_e^i \cup \{c\}$.

Therefore, the support for the least supported candidate is maximized if the total number of agents that support \mathcal{C}_j is shared out equally among all the candidates in \mathcal{C}_j that are in $\mathcal{C}_e^i \cup \{c\}$ (any other distribution of the votes

to \mathcal{C}_j would make one or several candidates in $(\mathcal{C}_e^i \cup \{c\}) \cap \mathcal{C}_j$ receive less support).

$$\max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\}) = \frac{B(\mathcal{C}_j)}{|(\mathcal{C}_e^i \cup \{c\}) \cap \mathcal{C}_j|} \quad (73)$$

This is exactly the same computation as that performed by the D'Hondt method for selecting candidates, so both methods must assign the same number of seats to each \mathcal{C}_j . \square

THEOREM 4 *ODH is house monotonic, that is, for any pair of elections $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and $\sigma_1 = \langle \mathcal{V}, \mathcal{C}, (S+1), B \rangle$, it holds that $\text{ODH}(\sigma) \subset \text{ODH}(\sigma_1)$.*

Proof.

Clearly, ODH is house monotonic due to the iterative process used to define the ODH presented in Algorithm 1 (selecting $(S+1)$ candidates, which comprise the first S candidates, and then running an additional iteration to select the last one). \square

Before proving that ODH satisfies the lower quota it is necessary to prove an additional lemma.

LEMMA 5 *For any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and any $i = 0, \dots, (S-1)$, let \mathcal{C}_e^i be the set of the first i candidates chosen by ODH (when $i = 0$, $\mathcal{C}_e^0 = \emptyset$).*

Consider a candidate $c \in \mathcal{C} - \mathcal{C}_e^i$ and a set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ such that $\forall j = 1, \dots, n; c \in y_j$. Then it holds that

$$\max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\}) \geq \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$$

Proof.

Suppose, for the sake of contradiction, that for a certain election σ , a certain candidate $c \in \mathcal{C} - \mathcal{C}_e^i$, and a certain set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ such that $\forall j = 1, \dots, n; c \in y_j$, it is $\max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\}) < \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$. Consider any support distribution function $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^i \cup \{c\})}^{\text{opt}}$. By Theorem 2, it should be $\text{Supp}_F(c) = \max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\})$, and therefore by the hypothesis it would be $\text{Supp}_F(c) < \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$.

Observe that 1) the total number of agents that have casted any of y_1, \dots, y_n is $\sum_{j=1}^n B(y_j)$; 2) the total number of candidates in $\mathcal{C}_e^i \cup \{c\}$ that have been approved by any of the agents that have casted any of y_1, \dots, y_n is $|(\mathcal{C}_e^i \cup \{c\}) \cap \bigcup_{j=1}^n y_j| = |\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1$; and 3) by the definition of support

distribution function (Equation (26)), the support distribution function F has to distribute all the votes of the agents that have approved any of the candidates in $\mathcal{C}_e^i \cup \{c\}$. But since $c \in y_j$, for all $j=1, \dots, n$, this means that all the votes of agents that have casted any of y_1, \dots, y_n must be distributed.

If F divided all the votes of agents that have casted any of y_1, \dots, y_n equally between the candidates in $(\mathcal{C}_e^i \cup \{c\}) \cap \bigcup_{j=1}^n y_j$, then the support of candidate c would be at least $\frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$ (recall that c could receive support also from other agents that have not casted any of y_1, \dots, y_n). This contradicts the hypothesis given at the beginning of the proof. Because the share of votes of the agents that have casted any of y_1, \dots, y_n assigned by F to candidate c is strictly smaller than $\frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$, a share of votes strictly greater than $\frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$ needs to be assigned by F to at least one of the candidates in $\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j$.

Let \mathcal{U} be the non-empty set of candidates such that F assigns them a share of the votes of agents that have casted any of y_1, \dots, y_n strictly greater than $\frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$.

$$\mathcal{U} = \{c^* \in \mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j, \sum_{j=1}^n F(y_j, c^*) > \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}\} \quad (74)$$

But since for all $j = 1, \dots, n$, it is $c \in y_j$, it is possible to transfer support to c received under F by the candidates in \mathcal{U} . The result would be a support distribution function F_1 such that:

1. for all candidates $c' \in \mathcal{C}_e^i - \mathcal{U}$, it is $Supp_{F_1}(c') = Supp_F(c')$,
2. $Supp_{F_1}(c) = \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$, and
3. for all candidates $c^* \in \mathcal{U}$, it is $Supp_{F_1}(c^*) \geq \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}$.

One possible way of defining F_1 is as follows. Let

$$\begin{aligned} k_1 &= \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}, \\ k_2(c^*) &= \sum_{j=1}^n F(y_j, c^*) \text{ for each } c^* \in \mathcal{U}, \\ k_3 &= \frac{\sum_{c^* \in \mathcal{U}} (k_2(c^*) - k_1)}{k_1 - Supp_F(c)} \geq 1, \text{ and} \\ k_4(c^*) &= \frac{k_2(c^*)}{k_1 + \frac{(k_2(c^*) - k_1)(k_3 - 1)}{k_3}} \text{ for each } c^* \in \mathcal{U}. \end{aligned} \quad (75)$$

Then, F_1 is defined as

$$\begin{aligned}
F_1(y, c') &= F(y, c') && \text{for each } y \in 2^{\mathcal{C}} - \{y_1, \dots, y_n\}, \\
&&& \text{for each } c' \in \mathcal{C}_e^i \cup \{c\} \\
F_1(y_j, c') &= F(y_j, c') && \text{for each } y_j \in \{y_1, \dots, y_n\}, \text{ for each } c' \in \mathcal{C}_e^i - \mathcal{U} \\
F_1(y_j, c^*) &= \frac{F(y_j, c^*)}{k_4(c^*)} && \text{for each } y_j \in \{y_1, \dots, y_n\}, \text{ for each } c^* \in \mathcal{U} \\
F_1(y_j, c) &= \sum_{c^* \in \mathcal{U}} (F(y_j, c^*) - F_1(y_j, c^*)) \\
&\quad + F(y_j, c) && \text{for each } y_j \in \{y_1, \dots, y_n\}
\end{aligned} \tag{76}$$

For all candidates $c^* \in \mathcal{U}$ we have

$$\begin{aligned}
\sum_{j=1}^n F_1(y_j, c^*) &= \sum_{j=1}^n \frac{F(y_j, c^*)}{k_4(c^*)} = \frac{\sum_{j=1}^n F(y_j, c^*)}{k_4(c^*)} = \frac{k_2(c^*)}{k_4(c^*)} \\
&= k_1 + \frac{(k_2(c^*) - k_1)(k_3 - 1)}{k_3} \geq k_1 = \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}.
\end{aligned} \tag{77}$$

For the candidate c we have

$$\begin{aligned}
Supp_{F_1}(c) &= Supp_F(c) + \sum_{j=1}^n \sum_{c^* \in \mathcal{U}} (F(y_j, c^*) - F_1(y_j, c^*)) \quad (78) \\
&= Supp_F(c) + \sum_{c^* \in \mathcal{U}} \sum_{j=1}^n (F(y_j, c^*) - F_1(y_j, c^*)) \\
&= Supp_F(c) + \sum_{c^* \in \mathcal{U}} \left(\sum_{j=1}^n F(y_j, c^*) - \sum_{j=1}^n F_1(y_j, c^*) \right) \\
&= Supp_F(c) + \sum_{c^* \in \mathcal{U}} \left(k_2(c^*) - \left(k_1 + \frac{(k_2(c^*) - k_1)(k_3 - 1)}{k_3} \right) \right) \\
&= Supp_F(c) + \sum_{c^* \in \mathcal{U}} \left((k_2(c^*) - k_1) \left(1 - \frac{k_3 - 1}{k_3} \right) \right) \\
&= Supp_F(c) + \sum_{c^* \in \mathcal{U}} \frac{k_2(c^*) - k_1}{k_3} \\
&= Supp_F(c) + \frac{1}{k_3} \sum_{c^* \in \mathcal{U}} (k_2(c^*) - k_1) \\
&= Supp_F(c) + \frac{k_1 - Supp_F(c)}{\sum_{c^* \in \mathcal{U}} (k_2(c^*) - k_1)} \sum_{c^* \in \mathcal{U}} (k_2(c^*) - k_1) \\
&= k_1 = \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^i \cap \bigcup_{j=1}^n y_j| + 1}.
\end{aligned}$$

Observe that all the candidates in \mathcal{C}_e^i have under F_1 either the same support that they had under F or at least the same support as c under F_1 .

Let \mathcal{L} be the set of the least supported candidates in $\{c\} \cup \mathcal{C}_e^i$ under F .

$$\mathcal{L} = \{\ell : \ell \in \{c\} \cup \mathcal{C}_e^i, Supp_F(\ell) = \max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\})\} \quad (79)$$

As we have already said, by Theorem 2 it is $c \in \mathcal{L}$. If $\mathcal{L} = \{c\}$ then the support of the least supported candidate under F_1 is strictly greater than the support of the least supported candidate under F , because the support of c under F_1 is strictly greater than the support of c under F , and all the candidates whose support under F_1 has decreased with respect to their support under F have still under F_1 at least the same support as c . This contradicts the hypothesis that $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^i \cup \{c\})}^{\text{opt}}$.

Otherwise, suppose that exist a candidate ℓ' different from c such that $\ell' \in \mathcal{L}$. Clearly, $\ell' \notin \mathcal{U}$, and therefore it is $Supp_{F_1}(\ell') = Supp_F(\ell') = \max\text{Min}(\sigma, \mathcal{C}_e^i \cup \{c\})$. This implies that $F_1 \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^i \cup \{c\})}^{\text{opt}}$ (again, recall that

all the candidates whose support under F_1 has decreased with respect to their support under F have still under F_1 at least the same support as c , and therefore the support under F_1 of the least supported candidate cannot be smaller than the support under F of the least supported candidate). Again we have reached a contradiction, because $Supp_{F_1}(c) = \frac{\sum_{j=1}^n B(y_j)}{|C_e^i \cap \bigcup_{j=1}^n y_j| + 1} > \max\text{Min}(\sigma, C_e^i \cup \{c\})$, and this would contradict Theorem 2.

We have seen that if we suppose that for certain support distribution function $F \in \mathfrak{F}_{\sigma, (C_e^i \cup \{c\})}^{\text{opt}}$ it is $Supp_F(c) = \max\text{Min}(\sigma, C_e^i \cup \{c\}) < \frac{\sum_{j=1}^n B(y_j)}{|C_e^i \cap \bigcup_{j=1}^n y_j| + 1}$, with $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ such that $\forall j = 1, \dots, n$, it is $c \in y_j$, we reach a contradiction. This proves that the lemma holds. \square

THEOREM 5 *ODH satisfies the lower quota. That is, for any given election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, candidate subset $\mathcal{A} \subseteq \mathcal{C}$ and set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ such that:*

$$\mathcal{A} \subseteq y_j, \forall j = 1, \dots, n \quad (80)$$

$$q = \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{V}|} S \quad (81)$$

$$|\mathcal{A}| \geq \lfloor q \rfloor \quad (82)$$

it is $|\text{ODH}(\sigma) \cap \bigcup_{j=1}^n y_j| \geq \lfloor q \rfloor$.

Proof.

Consider an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, a candidate subset $\mathcal{A} \subseteq \mathcal{C}$ and a set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ that verify the conditions established in the theorem.

For $h = 1, \dots, \lfloor q \rfloor$, let $\text{pos}(h)$ be

$$\text{pos}(h) = \left\lceil \frac{hS}{\lfloor q \rfloor} \right\rceil. \quad (83)$$

The proof is obtained as follows. For each $h = 1, \dots, \lfloor q \rfloor$, the assignment of the $\text{pos}(h)$ seat is considered. If at least h seats have already been assigned to candidates in $\bigcup_{j=1}^n y_j$, continue to $\text{pos}(h+1)$. Otherwise, it is assumed that $h-1$ candidates in $\bigcup_{j=1}^n y_j$ have already been assigned (thus, there is no assumption for $h=1$), and then it is proved that seat $\text{pos}(h)$ has to be assigned to a candidate in $\bigcup_{j=1}^n y_j$.

As before, let $C_e^{\text{pos}(h)-1}$ be the set of the first $\text{pos}(h)-1$ candidates chosen by the ODH. We partition the candidates in $\mathcal{C} - C_e^{\text{pos}(h)-1}$ in three subsets:

- Candidates in $\mathcal{A} - \mathcal{C}_e^{\text{pos}(h)-1}$. Observe that $\mathcal{A} - \mathcal{C}_e^{\text{pos}(h)-1}$ is non-empty, because $|\mathcal{A}| \geq \lfloor q \rfloor$ and $\mathcal{C}_e^{\text{pos}(h)-1} \cap \mathcal{A} \subseteq \mathcal{C}_e^{\text{pos}(h)-1} \cap \bigcup_{j=1}^n y_j$ and $|\mathcal{C}_e^{\text{pos}(h)-1} \cap \bigcup_{j=1}^n y_j| = h - 1 < \lfloor q \rfloor$.
- Candidates that do not belong to $\bigcup_{j=1}^n y_j$, that is, candidates in $\mathcal{C} - (\mathcal{C}_e^{\text{pos}(h)-1} \cup \bigcup_{j=1}^n y_j)$.
- The remaining candidates belong to $\bigcup_{j=1}^n y_j - (\mathcal{C}_e^{\text{pos}(h)-1} \cup \mathcal{A})$.

We are going to prove that, under the assumption that $|\mathcal{C}_e^{\text{pos}(h)-1} \cap \bigcup_{j=1}^n y_j| = h - 1$, then for each candidate $c_s \in \mathcal{A} - \mathcal{C}_e^{\text{pos}(h)-1}$ and for each candidate $c_{ns} \in \mathcal{C} - (\mathcal{C}_e^{\text{pos}(h)-1} \cup \bigcup_{j=1}^n y_j)$, it is $\text{maxMin}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_s\}) > \text{maxMin}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\})$. Therefore, the candidate elected at iteration $\text{pos}(h)$ cannot belong to $\mathcal{C} - (\mathcal{C}_e^{\text{pos}(h)-1} \cup \bigcup_{j=1}^n y_j)$ because the candidates in $\mathcal{A} \subseteq \bigcup_{j=1}^n y_j$ have a higher support. This proves that the candidate elected at iteration $\text{pos}(h)$ has to belong to $\bigcup_{j=1}^n y_j$, and thus that the theorem holds.

Consider any candidate $c_s \in \mathcal{A} - \mathcal{C}_e^{\text{pos}(h)-1}$. By Lemma 5, it is

$$\text{maxMin}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_s\}) \geq \frac{\sum_{j=1}^n B(y_j)}{|\mathcal{C}_e^{\text{pos}(h)-1} \cap \bigcup_{j=1}^n y_j| + 1} = \frac{\sum_{j=1}^n B(y_j)}{h} \quad (84)$$

Now, consider any candidate $c_{ns} \in \mathcal{C} - (\mathcal{C}_e^{\text{pos}(h)-1} \cup \bigcup_{j=1}^n y_j)$. Observe that: 1) for each support distribution function $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\})}^{\text{opt}}$, all candidates in $\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\}$ must have a support under F of at least $\text{maxMin}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\})$; 2) The number of candidates in $(\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\}) - \bigcup_{j=1}^n y_j$ is equal to $|\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\}| - (h - 1) = \text{pos}(h) + 1 - h$; and 3) the candidates in $\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\} - \bigcup_{j=1}^n y_j$ cannot receive any support from the agents that have casted any of y_1, \dots, y_n . Therefore, for any support distribution function $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\})}^{\text{opt}}$ it is

$$\begin{aligned} |\mathcal{V}| - \sum_{j=1}^n B(y_j) &\geq \sum_{c \in (\mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\}) - \bigcup_{j=1}^n y_j} \text{Supp}_F(c) \\ &\geq (\text{pos}(h) + 1 - h) \text{maxMin}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\}) \end{aligned} \quad (85)$$

and thus, taking into account Equation (85), the definition of $\text{pos}(h)$ (equation (83)) and the definition of q (equation (81)),

$$\begin{aligned}
\max\text{Min}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_{ns}\}) &\leq \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{\text{pos}(h) + 1 - h} \\
&< \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{\text{pos}(h) - h} = \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{\left\lceil \frac{hS}{[q]} \right\rceil - h} \\
&\leq \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{\frac{hS}{q} - h} = \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{h(\frac{S}{q} - 1)} \quad (86) \\
&= \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{h(\frac{|\mathcal{V}|}{\sum_{j=1}^n B(y_j)} - 1)} = \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{h \frac{|\mathcal{V}| - \sum_{j=1}^n B(y_j)}{\sum_{j=1}^n B(y_j)}} \\
&= \frac{\sum_{j=1}^n B(y_j)}{h} \leq \max\text{Min}(\sigma, \mathcal{C}_e^{\text{pos}(h)-1} \cup \{c_s\}).
\end{aligned}$$

□

THEOREM 6 *ODH is population monotonic, that is, for any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and for any non-empty candidate subset $\mathcal{G} \subseteq \text{ODH}(\sigma)$, the following conditions hold.*

1. *For any candidate subset $\mathcal{A} \subseteq \mathcal{C}$, such that $\mathcal{G} \cap \mathcal{A} = \emptyset$ and $B(\mathcal{A}) \geq 1$, consider the election $\sigma_1 = \langle \mathcal{V}_1, \mathcal{C}, S, B_1 \rangle$, where*

$$|\mathcal{V}_1| = |\mathcal{V}| \quad (87)$$

$$B_1(\mathcal{A}) = B(\mathcal{A}) - 1 \quad (88)$$

$$B_1(\mathcal{A} \cup \mathcal{G}) = B(\mathcal{A} \cup \mathcal{G}) + 1 \quad (89)$$

$$\begin{aligned}
\forall \mathcal{X} \in 2^{\mathcal{C}}, \mathcal{X} \neq \mathcal{A}, \mathcal{X} \neq (\mathcal{A} \cup \mathcal{G}) \Rightarrow \\
B_1(\mathcal{X}) = B(\mathcal{X}), \quad (90)
\end{aligned}$$

and thus it must hold that $\mathcal{G} \cap \text{ODH}(\sigma_1) \neq \emptyset$.

2. *Consider the election $\sigma_2 = \langle \mathcal{V}_2, \mathcal{C}, S, B_2 \rangle$, where*

$$|\mathcal{V}_2| = |\mathcal{V}| + 1 \quad (91)$$

$$B_2(\mathcal{G}) = B(\mathcal{G}) + 1 \quad (92)$$

$$\forall \mathcal{X} \in 2^{\mathcal{C}}, \mathcal{X} \neq \mathcal{G} \Rightarrow B_2(\mathcal{X}) = B(\mathcal{X}), \quad (93)$$

and thus it must hold that $\mathcal{G} \cap ODH(\sigma_2) \neq \emptyset$.

Proof.

We are going to show that for a candidate subset \mathcal{X} disjoint from \mathcal{G} , it is possible to define 1) an application g from $\mathfrak{F}_{\sigma, \mathcal{X}}$ to $\mathfrak{F}_{\sigma_1, \mathcal{X}}$ such that for each support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{X}}$ and for each candidate $c \in \mathcal{X}$ it is $Supp_F(c) = Supp_{g(F)}(c)$; and 2) an application g_1 from $\mathfrak{F}_{\sigma_1, \mathcal{X}}$ to $\mathfrak{F}_{\sigma, \mathcal{X}}$ such that for each support distribution function $F_1 \in \mathfrak{F}_{\sigma_1, \mathcal{X}}$ and for each candidate $c \in \mathcal{X}$ it is $Supp_{F_1}(c) = Supp_{g_1(F_1)}(c)$. This implies that for any support distribution functions $F^{\text{opt}} \in \mathfrak{F}_{\sigma, \mathcal{X}}^{\text{opt}}$ and $F_1^{\text{opt}} \in \mathfrak{F}_{\sigma_1, \mathcal{X}}^{\text{opt}}$ it is

$$\begin{aligned} \max\text{Min}(\sigma, \mathcal{X}) &= \min_{c \in \mathcal{X}} Supp_{F^{\text{opt}}}(c) = \min_{c \in \mathcal{X}} Supp_{g(F^{\text{opt}})}(c) \\ &\leq \max\text{Min}(\sigma_1, \mathcal{X}) = \min_{c \in \mathcal{X}} Supp_{F_1^{\text{opt}}}(c) = \min_{c \in \mathcal{X}} Supp_{g_1(F_1^{\text{opt}})}(c) \\ &\leq \max\text{Min}(\sigma, \mathcal{X}), \end{aligned} \quad (94)$$

and therefore it is $\max\text{Min}(\sigma, \mathcal{X}) = \max\text{Min}(\sigma_1, \mathcal{X})$.

For each support distribution function $F \in \mathfrak{F}_{\sigma, \mathcal{X}}$, $g(F)$ simply has to distribute one unit of support from the agents that approve $\mathcal{A} \cup \mathcal{G}$ as does F for the agents that approve \mathcal{A} :

$$g(F)(\mathcal{A}, c) = F(\mathcal{A}, c) \frac{B(\mathcal{A})-1}{B(\mathcal{A})} \quad \text{for each } c \in \mathcal{X} \quad (95)$$

$$g(F)(\mathcal{A} \cup \mathcal{G}, c) = F(\mathcal{A} \cup \mathcal{G}, c) + \frac{F(\mathcal{A}, c)}{B(\mathcal{A})} \quad \text{for each } c \in \mathcal{X} \quad (96)$$

$$g(F)(y, c) = F(y, c) \quad \text{for each other } (y, c) \quad (97)$$

For each support distribution function $F_1 \in \mathfrak{F}_{\sigma_1, \mathcal{X}}$, $g_1(F_1)$ simply has to distribute one unit of support from the agents that approve \mathcal{A} as does F_1 for the agents that approve $\mathcal{A} \cup \mathcal{G}$:

$$g_1(F_1)(\mathcal{A}, c) = F_1(\mathcal{A}, c) + \frac{F_1(\mathcal{A} \cup \mathcal{G}, c)}{B_1(\mathcal{A} \cup \mathcal{G})} \quad \text{for each } c \in \mathcal{X} \quad (98)$$

$$g_1(F_1)(\mathcal{A} \cup \mathcal{G}, c) = F_1(\mathcal{A} \cup \mathcal{G}, c) \frac{B_1(\mathcal{A} \cup \mathcal{G})-1}{B_1(\mathcal{A} \cup \mathcal{G})} \quad \text{for each } c \in \mathcal{X} \quad (99)$$

$$g_1(F_1)(y, c) = F_1(y, c) \quad \text{for each other } (y, c) \quad (100)$$

Observe that since \mathcal{X} is disjoint from \mathcal{G} , the candidates from \mathcal{X} that can receive support from agents that approve the candidates in \mathcal{A} are the same as the candidates from \mathcal{X} that can receive support from agents that approve the candidates in $\mathcal{A} \cup \mathcal{G}$.

In the case of election σ_2 the situation is easier, because as no candidate in \mathcal{X} is approved by the agents that approve \mathcal{G} , for any support distribution function $F \in \mathfrak{F}_{\sigma_2, \mathcal{X}}$ it has to be $F(\mathcal{G}, c) = 0$ for each candidate $c \in \mathcal{X}$, and therefore it is $\mathfrak{F}_{\sigma, \mathcal{X}} = \mathfrak{F}_{\sigma_2, \mathcal{X}}$ and thus $\max\text{Min}(\sigma, \mathcal{X}) = \max\text{Min}(\sigma_2, \mathcal{X})$.

Let i be the ODH iteration in which the first candidate from \mathcal{G} is elected in election σ and let $c \in \mathcal{G}$ be such candidate. Let $\mathcal{C}_{e\sigma}^j$ (respectively $\mathcal{C}_{e\sigma_1}^j$ and $\mathcal{C}_{e\sigma_2}^j$ for elections σ_1 and σ_2) be the first j candidates chosen by ODH for election σ . For $j = 0$ it is $\mathcal{C}_{e\sigma}^0 = \mathcal{C}_{e\sigma_1}^0 = \mathcal{C}_{e\sigma_2}^0 = \emptyset$.

For the first $(i - 1)$ iterations of the execution of ODH for elections σ_1 and σ_2 there are two possibilities:

1. at least one candidate from \mathcal{G} is elected (in that case, the theorem holds);
2. otherwise, if for all $j = 1, \dots, (i - 1)$ it is $\mathcal{C}_{e\sigma_1}^j \cap \mathcal{G} = \emptyset$ (respectively $\mathcal{C}_{e\sigma_2}^j \cap \mathcal{G} = \emptyset$), then for all $j = 1, \dots, (i - 1)$ and for any not yet elected candidate c' that does not belong to \mathcal{G} , it would be $(\mathcal{C}_{e\sigma_1}^j \cup \{c'\}) \cap \mathcal{G} = \emptyset$ (respectively $(\mathcal{C}_{e\sigma_2}^j \cup \{c'\}) \cap \mathcal{G} = \emptyset$), and therefore, as we have just seen, it would be $\max\text{Min}(\sigma_1, \mathcal{C}_{e\sigma_1}^j \cup \{c'\}) = \max\text{Min}(\sigma, \mathcal{C}_{e\sigma_1}^j \cup \{c'\})$ (respectively $\max\text{Min}(\sigma_2, \mathcal{C}_{e\sigma_2}^j \cup \{c'\}) = \max\text{Min}(\sigma, \mathcal{C}_{e\sigma_2}^j \cup \{c'\})$).

But therefore, at each iteration $j = 1, \dots, (i - 1)$ the elected candidate would be the same for elections σ , σ_1 and σ_2 , and thus it would be $\mathcal{C}_{e\sigma}^j = \mathcal{C}_{e\sigma_1}^j$ (respectively $\mathcal{C}_{e\sigma}^j = \mathcal{C}_{e\sigma_2}^j$), and in particular it would be $\mathcal{C}_{e\sigma}^{i-1} = \mathcal{C}_{e\sigma_1}^{i-1}$ (respectively $\mathcal{C}_{e\sigma}^{i-1} = \mathcal{C}_{e\sigma_2}^{i-1}$).

Assuming that no candidate from \mathcal{G} has been elected in the first $(i - 1)$ iterations, consider any support distribution function $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_{e\sigma}^{i-1} \cup \{c\})}^{\text{opt}}$.

For election σ_1 it is possible to build a support distribution function $F_1 \in \mathfrak{F}_{\sigma_1, (\mathcal{C}_{e\sigma}^{i-1} \cup \{c\})}$ that assigns each candidate in $\mathcal{C}_{e\sigma}^{i-1}$ with the same support as F , and c with at least the same support as F . The only difference between σ and σ_1 is that one agent a votes for \mathcal{A} in σ and for $\mathcal{A} \cup \mathcal{G}$ in σ_1 . For all of the agents that have not changed their vote, F_1 distributes their votes in exactly the same way as F . For the agent a that voted for \mathcal{A} in σ and for $\mathcal{A} \cup \mathcal{G}$ in σ_1 , there are two possibilities:

- If any of the candidates in \mathcal{A} have already been elected, i.e., if $\mathcal{C}_{e\sigma}^{i-1} \cap \mathcal{A} \neq \emptyset$ (observe that we are assuming that $\mathcal{C}_{e\sigma}^{i-1} \cap \mathcal{G} = \emptyset$, but we do not make any assumption about $\mathcal{C}_{e\sigma}^{i-1} \cap \mathcal{A}$), then F has to distribute the vote of agent a between the candidates in $\mathcal{C}_{e\sigma}^{i-1} \cap \mathcal{A}$. In this case, F_1 distributes the vote of a between the candidates in $\mathcal{C}_{e\sigma}^{i-1} \cap \mathcal{A}$ in exactly the same way as F . Therefore, all of the candidates in $\mathcal{C}_{e\sigma}^{i-1} \cup \{c\}$ receive the same support with F_1 as that with F .

- Otherwise, since no candidate in \mathcal{A} has already been elected, then F cannot distribute votes to \mathcal{A} for any candidate in $\mathcal{C}_{e\sigma}^{i-1} \cup \{c\}$. F_1 has to distribute the vote of agent a to c . In this case, all of the candidates in $\mathcal{C}_{e\sigma}^{i-1}$ receive the same support with F_1 as with F , and c receives one vote more with F_1 than with F .

This is illustrated with election σ^a , that was presented in section 4.2, and for which the output produced by ODH was computed in section 6. We give again the result of election σ^a in Table 7. Table 7 shows also the result of election σ^{a1} , which is also used to illustrate the proof. Election σ^{a1} differs from election σ^a in that one agent chooses to approve candidates $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$ instead of approving only $\{\mathbf{a}, \mathbf{b}\}$.

As it can be seen, election σ^{a1} is a modification of election σ^a in the way defined by the first condition of the theorem. In this example we have $\mathcal{G} = \{\mathbf{d}\}$ and $\mathcal{A} = \{\mathbf{a}, \mathbf{b}\}$.

Candidates	Votes in σ^a (B^1)	Votes in σ^{a1} (B^{a1})
$\{\mathbf{a}, \mathbf{b}\}$	10,000	9,999
$\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$	0	1
$\{\mathbf{a}, \mathbf{c}\}$	6,000	6,000
$\{\mathbf{b}\}$	4,000	4,000
$\{\mathbf{c}\}$	5,500	5,500
$\{\mathbf{d}\}$	9,500	9,500
$\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$	5,000	5,000
$\{\mathbf{e}\}$	3,000	3,000

Table 7: Election result for σ^a and σ^{a1}

For election σ^a we have $i = 3$ (because candidate \mathbf{d} is elected in election σ^a at iteration 3, see section 6), $\mathcal{C}_{e\sigma^a}^{i-1} = \mathcal{C}_{e\sigma^a}^2 = \{\mathbf{a}, \mathbf{c}\}$, $\mathcal{C}_{e\sigma^a}^2 \cup \{\mathbf{d}\} = \{\mathbf{a}, \mathbf{c}, \mathbf{d}\}$. Consider the support distribution function $F^{a,\text{opt}} \in \mathfrak{F}_{\sigma^a, \{\mathbf{a}, \mathbf{c}, \mathbf{d}\}}^{\text{opt}}$, defined as:

$$\begin{aligned}
F^{a,\text{opt}}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) &= 10,000 \\
F^{a,\text{opt}}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{a}) &= 750 \\
F^{a,\text{opt}}(\{\mathbf{a}, \mathbf{c}\}, \mathbf{c}) &= 5,250 \\
F^{a,\text{opt}}(\{\mathbf{c}\}, \mathbf{c}) &= 5,500 \\
F^{a,\text{opt}}(\{\mathbf{d}\}, \mathbf{d}) &= 9,500
\end{aligned}$$

For election σ^{a1} the support distribution function $F_1^{a1} \in \mathfrak{F}_{\sigma^{a1}, \{\mathbf{a}, \mathbf{c}, \mathbf{d}\}}$, defined from $F^{a,\text{opt}}$ would be:

$$\begin{aligned}
F_1^{a1}(\{\mathbf{a}, \mathbf{b}\}, \mathbf{a}) &= 9,999 \\
F_1^{a1}(\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}, \mathbf{a}) &= 1 \\
\text{otherwise, } F_1^{a1}(y, c') &= F^{a, \text{opt}}(y, c')
\end{aligned}$$

Continuing with the formal proof, since F_1 distributes each candidate in $\mathcal{C}_{e\sigma}^{i-1} \cup \{c\}$ with at least the same support as F , and $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_{e\sigma}^{i-1} \cup \{c\})}^{\text{opt}}$, then it follows that

$$\forall c' \in \mathcal{C}_{e\sigma}^{i-1}, \text{Supp}_{F_1}(c') = \text{Supp}_F(c') \quad (101)$$

$$\text{Supp}_{F_1}(c) \geq \text{Supp}_F(c) \quad (102)$$

$$\begin{aligned}
\max\text{Min}(\sigma_1, \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}) &\geq \min_{c' \in \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}} \text{Supp}_{F_1}(c') \geq \min_{c' \in \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}} \text{Supp}_F(c') \\
&= \max\text{Min}(\sigma, \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}).
\end{aligned} \quad (103)$$

For any other unelected candidate $c'' \in \mathcal{C} - (\mathcal{C}_{e\sigma}^{i-1} \cup \mathcal{G})$, it holds that $\max\text{Min}(\sigma_1, \mathcal{C}_{e\sigma}^{i-1} \cup \{c''\}) = \max\text{Min}(\sigma, \mathcal{C}_{e\sigma}^{i-1} \cup \{c''\})$, and because c is the candidate elected by the ODH at iteration i for election σ , then the candidates in $\mathcal{C} - (\mathcal{C}_{e\sigma}^{i-1} \cup \mathcal{G})$ cannot be elected at iteration i for election σ_1 . This proves that the candidate elected at iteration i for election σ_1 must belong to \mathcal{G} .

Similarly, for any support distribution function $F \in \mathfrak{F}_{\sigma, (\mathcal{C}_{e\sigma}^{i-1} \cup \{c\})}^{\text{opt}}$, it is possible to build a support distribution function $F_2 \in \mathfrak{F}_{\sigma_2, (\mathcal{C}_{e\sigma}^{i-1} \cup \{c\})}$ that assigns each candidate in $\mathcal{C}_{e\sigma}^{i-1}$ with the same support as F and that assigns c with one unit of support plus the support that F assigned her. The only difference between σ_2 and σ is that in σ_2 a new agent b enters the election who approves the candidates in \mathcal{G} . For all the agents that participated in election σ , F_2 distributes their votes in exactly the same way as F , while F_2 assigns the vote of agent b to candidate c (observe that the only candidate from \mathcal{G} that appears in $\mathcal{C}_{e\sigma}^{i-1} \cup \{c\}$ is c).

The same discussion applies as that given in the previous case. F_2 distributes each candidate in $\mathcal{C}_{e\sigma}^{i-1}$ with the same support share as F , but with a higher support for c . Then,

$$\forall c' \in \mathcal{C}_{e\sigma}^{i-1}, \text{Supp}_{F_2}(c') = \text{Supp}_F(c') \quad (104)$$

$$\text{Supp}_{F_2}(c) > \text{Supp}_F(c) \quad (105)$$

$$\begin{aligned}
\max\text{Min}(\sigma_2, \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}) &\geq \min_{c' \in \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}} \text{Supp}_{F_2}(c') \geq \min_{c' \in \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}} \text{Supp}_F(c') \\
&= \max\text{Min}(\sigma, \mathcal{C}_{e\sigma}^{i-1} \cup \{c\}).
\end{aligned} \quad (106)$$

Again, as discussed earlier, for any other unelected candidate $c'' \in \mathcal{C} - (\mathcal{C}_{e\sigma}^{i-1} \cup \mathcal{G})$, it holds that $\max\text{Min}(\sigma_2, \mathcal{C}_{e\sigma}^{i-1} \cup \{c''\}) = \max\text{Min}(\sigma, \mathcal{C}_{e\sigma}^{i-1} \cup \{c''\})$, and because c is the candidate elected by the ODH at iteration i for election σ , then the candidates in $\mathcal{C} - (\mathcal{C}_{e\sigma}^{i-1} \cup \mathcal{G})$ cannot be elected at iteration i for election σ_2 . This proves that the candidate elected at iteration i for election σ_2 must belong to \mathcal{G} . \square

8 Comparison with other methods

We have already said that we believe that it is impossible to prove that certain approval-based multi-winner voting rule is the “best” possible extension to the D’Hondt method. On the other hand, it is possible to evaluate how good is an extension to the D’Hondt method for approval-based multi-winner elections (that is, an approval-based multi-winner voting rule that is equivalent to D’Hondt under closed lists) with respect to the properties that we have defined in this paper: house monotonicity, lower quota satisfaction, and population monotonicity. Moreover, it is also possible to evaluate these properties even for other approval-based multi-winner voting rules, that are not equivalent to D’Hondt under closed lists.

In this section we are going to study several approval-based multi-winner voting rules. We start with the second voting rule that we propose in this paper: an alternative to ODH that instead of using an iterative method outputs the set of winners that globally maximizes the support of the least supported winner. We call this the optimal ODH or OODH. Next, we will study the other two approval-based multi-winner voting rules that we are aware that are equivalent to D’Hondt under closed lists: the Reweighted Approval Voting (RAV) and the Proportional Approval Voting (PAV), surveyed by Kilgour [31]. Finally, we will study other approval-based multi-winner voting rules also surveyed by Kilgour [31] and the approval-based variants of the Chamberlin and Courant [12] and Monroe [35] voting rules.

We will show that 1) none of these voting rules at the same time is house monotonic, satisfies the lower quota, and is population monotonic; and 2) none of the voting rules studied that can be computed in polynomial time satisfies the lower quota. For the sake of brevity, in most cases we will simply give counterexamples that show that the statement made in the previous sentence is correct. An exhaustive study of the properties of these voting rules is left as future work.

We will take advantage of previous results by Aziz et al. [2]. In [2] two axioms are defined that are related to our lower quota property. They are called *justified representation* (JR) and *extended justified representation* (EJR). In

the context of the model for approval-based multi-winner elections that we use in this paper they can be defined as follows.

Justified representation (JR). Given an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, we say that a set of candidates $\mathcal{W} \subseteq \mathcal{C}$ of size $|\mathcal{W}| = S$ provides justified representation for σ if there does not exist a set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ such that $B(y_i) > 0$ for each y_i , $\sum_{i=1}^n B(y_i) \geq \frac{|\mathcal{V}|}{S}$, $\bigcap_{i=1}^n y_i \neq \emptyset$ and $\mathcal{W} \cap \bigcup_{i=1}^n y_i = \emptyset$. We say that an approval-based multi-winner voting rule satisfies justified representation (JR) if for every approval-based multi-winner election σ it outputs a winning set that provides justified representation for σ .

Extended justified representation (EJR). Given an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, we say that a set of candidates $\mathcal{W} \subseteq \mathcal{C}$ of size $|\mathcal{W}| = S$ provides extended justified representation for σ if there does not exist a set of candidate subsets $\{y_1, \dots, y_n\} \subseteq 2^{\mathcal{C}}$ and a positive integer $\ell \leq S$ such that $B(y_i) > 0$ for each y_i , $\sum_{i=1}^n B(y_i) \geq \ell \frac{|\mathcal{V}|}{S}$, $|\bigcap_{i=1}^n y_i| \geq \ell$ and $|y_i \cap \mathcal{W}| < \ell$ for each $i = 1, \dots, n$. We say that an approval-based multi-winner voting rule satisfies extended justified representation (EJR) if for every approval-based multi-winner election σ it outputs a winning set that provides extended justified representation for σ .

The following two lemmas were proved by Sánchez Fernández *et al.* in [41].

LEMMA 6 *If a voting rule satisfies the lower quota, then it has also to satisfy justified representation.*

LEMMA 7 *If a voting rule satisfies extended justified representation, then it has also to satisfy the lower quota.*

8.1 The optimal ODH method or OODH

As we said before, the idea of OODH is to output the set of winners that globally allow to maximize the support of the least supported winner. Formally, for any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, the output of OODH for election σ is defined as

$$OODH(\sigma) = \operatorname{argmax}_{\mathcal{W} \subseteq \mathcal{C}, |\mathcal{W}|=S} \max \operatorname{Min}(\sigma, \mathcal{W}) \quad (107)$$

There exist elections σ for which OODH provides with “better” sets of winners than ODH in the sense that $\max \operatorname{Min}(\sigma, OODH(\sigma))$ can be strictly greater than $\max \operatorname{Min}(\sigma, ODH(\sigma))$. A couple of examples will be shown later. However, OODH has a crucial disadvantage compared with ODH: it cannot be computed in polynomial time.

THEOREM 7 *If $P \neq NP$, then OODH cannot be computed in polynomial time.*

Proof.

We borrow a proof by Procaccia *et al.* [39]. Following the ideas exposed in that proof we show a polynomial-time reduction from the EXACT 3-COVER (X3C) problem to the problem of determining a winning set of candidates with OODH. The X3C problem is known to be NP-Complete [25], and therefore such polynomial-time reduction proves the theorem.

First, we repeat here the definition of the X3C problem given in [39].

In the X3C problem we are given a set \mathcal{U} of n points such that n is divisible by 3, and a collection of r subsets of \mathcal{U} , $\mathcal{F} = \{F_1, \dots, F_r\}$, each of cardinality 3, i.e., for all j , $|F_j| = 3$. We are asked whether it is possible to find $n/3$ disjoint subsets in \mathcal{F} such that their union covers the entire set \mathcal{U} .

Given an instance of the X3C problem, we map it to an approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, as follows. \mathcal{U} is the set of agents (that is, $\mathcal{V} = \mathcal{U}$), \mathcal{F} is the set of candidates (that is, $\mathcal{C} = \mathcal{F}$), and $S = n/3$. Each agent $u \in \mathcal{U}$ approves all the candidates F_i such that $u \in F_i$ in the original instance of the X3C problem. Observe that according to the definition of X3C $|\mathcal{V}|/S$ is the positive integer 3.

Let \mathcal{W} be the winning set of candidates for σ with OODH. In the case that $\max\text{Min}(\sigma, \mathcal{W}) = 3$, then all the candidates in the winning set have a support of at least 3. But since each candidate is approved only by three agents (in other words, each F_i contains only 3 points) this means that all the candidates in the winning set have a support equal to 3, and that all the agents are represented by a candidate that they approve. No agent can share its support between two or more candidates because in that case those candidates cannot have 3 units of support (each candidate is approved only by three agents). Assigning each candidate in the winning set to the agents that approve her solves the original X3C problem.

Conversely, if the X3C problem has a solution, such solution would be a set of winners \mathcal{W} such that $\max\text{Min}(\sigma, \mathcal{W}) = 3$, because in that case each agent approves one and only one of the candidates in the set of winners (the subsets in the solution cover \mathcal{U} and are disjoint) and each winner is approved by 3 agents. \square

OODH, like ODH, is equivalent to D'Hondt under closed lists.

THEOREM 8 *OODH is equivalent to D'Hondt under closed lists.*

Proof.

We have already discussed that the original D'Hondt method possesses the property of maximizing the support of the least supported elected candidate [18].

Since OODH also maximizes the support of the least supported elected candidate, when an approval-based multi-winner election σ fulfills the conditions established in the equivalent to D'Hondt under closed lists property (see section 4.3) OODH has to select the same number of winners from each set of candidates \mathcal{C}_i , equivalent to a list, as it does D'Hondt in the equivalent closed lists election. \square

Surprisingly enough, OODH fails the lower quota.

THEOREM 9 *OODH fails the lower quota.*

Proof.

Consider the election $\sigma^c = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$. Six agents ($|\mathcal{V}| = 6$) have to choose four candidates ($S = 4$) from $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5\}$. We have $B(\{c_1\}) = 1, B(\{c_2\}) = 1, B(\{c_3\}) = 1$, and $B(\{c_4, c_5\}) = 3$. According to the lower quota property both c_4 and c_5 must be elected.

Consider any subset \mathcal{W} of \mathcal{C} composed of c_4, c_5 , and any two of the remaining candidates. Clearly, $\max\text{Min}(\sigma, \mathcal{W}) = 1$ because c_1, c_2 , and c_3 cannot have a support greater than 1. But $\max\text{Min}(\sigma, \{c_1, c_2, c_3, c_4\})$ is also equal to 1. This proves that $\{c_1, c_2, c_3, c_4\}$ can be the set of winners with the OODH voting rule, depending on how ties are broken. \square

Observe that in the case of ODH the set of winners for election σ^c has always to include c_4 and c_5 (in fact, they are elected in the first two iterations of ODH).

Whether OODH satisfies the lower quota if ties are broken in favor of sets of winners that fulfill the conditions imposed by the lower quota is left open.

THEOREM 10 *OODH is population monotonic, that is, for any approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$ and for any non-empty candidate subset $\mathcal{G} \subseteq \text{OODH}(\sigma)$, the following conditions hold.*

1. *For any candidate subset $\mathcal{A} \subseteq \mathcal{C}$, such that $\mathcal{G} \cap \mathcal{A} = \emptyset$ and $B(\mathcal{A}) \geq 1$, consider the election $\sigma_1 = \langle \mathcal{V}_1, \mathcal{C}, S, B_1 \rangle$, where*

$$|\mathcal{V}_1| = |\mathcal{V}| \tag{108}$$

$$B_1(\mathcal{A}) = B(\mathcal{A}) - 1 \tag{109}$$

$$B_1(\mathcal{A} \cup \mathcal{G}) = B(\mathcal{A} \cup \mathcal{G}) + 1 \tag{110}$$

$$\forall \mathcal{X} \in 2^{\mathcal{C}}, \mathcal{X} \neq \mathcal{A}, \mathcal{X} \neq (\mathcal{A} \cup \mathcal{G}) \Rightarrow B_1(\mathcal{X}) = B(\mathcal{X}), \tag{111}$$

and thus it must hold that $\mathcal{G} \cap \text{OODH}(\sigma_1) \neq \emptyset$.

2. Consider the election $\sigma_2 = \langle \mathcal{V}_2, \mathcal{C}, S, B_2 \rangle$, where

$$|\mathcal{V}_2| = |\mathcal{V}| + 1 \quad (112)$$

$$B_2(\mathcal{G}) = B(\mathcal{G}) + 1 \quad (113)$$

$$\forall \mathcal{X} \in 2^{\mathcal{C}}, \mathcal{X} \neq \mathcal{G} \Rightarrow B_2(\mathcal{X}) = B(\mathcal{X}), \quad (114)$$

and thus it must hold that $\mathcal{G} \cap \text{OODH}(\sigma_2) \neq \emptyset$.

Proof.

The proof is similar to the equivalent proof for ODH. Since $\text{OODH}(\sigma)$ is the set of winners for election σ , it must be $\max\text{Min}(\sigma, \text{OODH}(\sigma)) \geq \max\text{Min}(\sigma, \mathcal{X})$ for any candidate subset $\mathcal{X} \subseteq \mathcal{C}, |\mathcal{X}| = S$. We have seen in theorem 6 that for a candidate subset \mathcal{X} disjoint from \mathcal{G} it is $\max\text{Min}(\sigma, \mathcal{X}) = \max\text{Min}(\sigma_1, \mathcal{X}) = \max\text{Min}(\sigma_2, \mathcal{X})$. Therefore, when $\mathcal{X} \cap \mathcal{G} = \emptyset$ and $|\mathcal{X}| = S$ it should be $\max\text{Min}(\sigma, \text{OODH}(\sigma)) \geq \max\text{Min}(\sigma_1, \mathcal{X})$ and $\max\text{Min}(\sigma, \text{OODH}(\sigma)) \geq \max\text{Min}(\sigma_2, \mathcal{X})$. Therefore, to prove the theorem it is enough to show that $\max\text{Min}(\sigma_1, \text{OODH}(\sigma)) \geq \max\text{Min}(\sigma, \text{OODH}(\sigma))$ and that $\max\text{Min}(\sigma_2, \text{OODH}(\sigma)) \geq \max\text{Min}(\sigma, \text{OODH}(\sigma))$.

Consider any support distribution function $F \in \mathfrak{F}_{\sigma, \text{OODH}(\sigma)}^{\text{opt}}$. For election σ_1 it is possible to build a support distribution function $F_1 \in \mathfrak{F}_{\sigma_1, \text{OODH}(\sigma)}$ that assigns each candidate in $\text{OODH}(\sigma) - \mathcal{G}$ with the same support as F , and to the candidates in \mathcal{G} with at least the same support as F . The only difference between σ and σ_1 is that one agent a votes for \mathcal{A} in σ and for $\mathcal{A} \cup \mathcal{G}$ in σ_1 . For all of the agents that have not changed their vote, F_1 distributes their votes in exactly the same way as F . For the agent a that voted for \mathcal{A} in σ and for $\mathcal{A} \cup \mathcal{G}$ in σ_1 , there are two possibilities:

- If any of the candidates in \mathcal{A} is a winner for election σ (that is, if $\mathcal{A} \cap \text{OODH}(\sigma) \neq \emptyset$), then F has to distribute the vote of agent a between the candidates in $\mathcal{A} \cap \text{OODH}(\sigma)$. In this case, F_1 distributes the vote of a between the candidates in $\mathcal{A} \cap \text{OODH}(\sigma)$ in exactly the same way as F . Therefore, all of the candidates in $\text{OODH}(\sigma)$ receive the same support with F_1 as with F .
- Otherwise, since no candidate in \mathcal{A} is in the set of winners for election σ , then F cannot distribute votes to \mathcal{A} for any candidate in $\text{OODH}(\sigma)$. F_1 has to distribute the vote of agent a to the candidates in \mathcal{G} . We can do that in any way we like. In this case, all of the candidates in

OODH(σ) – \mathcal{G} receive the same support with F_1 as with F , and each candidate in \mathcal{G} receive a support with F_1 greater than or equal to the support that such candidate receives with F .

Since F_1 distributes each candidate in OODH(σ) with at least the same support as F , and $F \in \mathfrak{F}_{\sigma, \text{OODH}(\sigma)}^{\text{opt}}$, then it follows that

$$\forall c \in \text{OODH}(\sigma), \text{ it is } \text{Supp}_{F_1}(c) \geq \text{Supp}_F(c) \quad (115)$$

$$\begin{aligned} \max\text{Min}(\sigma_1, \text{OODH}(\sigma)) &\geq \min_{\forall c \in \text{OODH}(\sigma)} \text{Supp}_{F_1}(c) \geq \min_{\forall c \in \text{OODH}(\sigma)} \text{Supp}_F(c) \\ &= \max\text{Min}(\sigma, \text{OODH}(\sigma)). \end{aligned} \quad (116)$$

Similarly, for any support distribution function $F \in \mathfrak{F}_{\sigma, \text{OODH}(\sigma)}^{\text{opt}}$, it is possible to build a support distribution function $F_2 \in \mathfrak{F}_{\sigma_2, \text{OODH}(\sigma)}$ so that each candidate in OODH(σ) – \mathcal{G} receives the same support with F_2 as with F and each candidate in \mathcal{G} receives a support with F_2 greater than or equal to the support that such candidate receives with F . The only difference between σ_2 and σ is that in σ_2 a new agent b enters the election who approves the candidates in \mathcal{G} . For all the agents that participated in election σ , F_2 distributes their votes in exactly the same way as F , while F_2 assigns the vote of agent b to the candidates in \mathcal{G} in any way we like.

The same discussion applies as that given in the previous case. F_2 distributes each candidate in OODH(σ) with at least the same support as F . Then,

$$\forall c \in \text{OODH}(\sigma), \text{Supp}_{F_2}(c) \geq \text{Supp}_F(c) \quad (117)$$

$$\begin{aligned} \max\text{Min}(\sigma_2, \text{OODH}(\sigma)) &\geq \min_{\forall c \in \text{OODH}(\sigma)} \text{Supp}_{F_2}(c) \geq \min_{\forall c \in \text{OODH}(\sigma)} \text{Supp}_F(c) \\ &= \max\text{Min}(\sigma, \text{OODH}(\sigma)). \end{aligned} \quad (118)$$

□

8.2 Other voting rules that are equivalent to D'Hondt under closed lists

In this section we study two voting rules that are surveyed by Kilgour [31]: the Reweighted Approval Voting and the Proportional Approval Voting.

Reweighted Approval Voting (RAV). RAV uses an iterative algorithm to select candidates. At each iteration one candidate is added to the set \mathcal{W} of elected candidates. Let \mathcal{W}_i be the set of candidates that have

already been elected after the i -th iteration. For each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, the *approval weight* (aw) at iteration i of each not yet elected candidate c in $(\mathcal{C} - \mathcal{W}_{i-1})$ is computed as:

$$\text{aw}(c) = \sum_{\mathcal{A} \subseteq \mathcal{C}, c \in \mathcal{A}} \frac{B(\mathcal{A})}{1 + |\mathcal{A} \cap \mathcal{W}_{i-1}|} \quad (119)$$

Then, at each iteration the candidate with highest approval weight is selected. The following lemma is immediate.

LEMMA 8 *RAV is equivalent to D'Hondt under closed lists.*

It is also straightforward to show that RAV can be computed in polynomial time [3]. Aziz *et al.* [2] proved that that RAV fails JR if $S \geq 10$. This result has been recently improved by Sánchez Fernández *et al.* [41] that show that RAV satisfies JR if $S \leq 5$ but fails it for $S \geq 6$. Therefore,

LEMMA 9 *RAV fails the lower quota.*

Proportional Approval Voting (PAV). Under PAV, each agent's utility $r(p)$ is $\sum_{j=1}^p \frac{1}{j}$, where p is the number of candidates approved by such agent that have been elected. PAV elects the set of candidates that maximize the sum of the PAV utilities for each agent (PAV-score). Formally, for each fully open election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$:

$$PAV(\sigma) = \underset{\mathcal{W} \subseteq \mathcal{C}, |\mathcal{W}|=S}{\operatorname{argmax}} \sum_{\mathcal{A} \subseteq \mathcal{C}} B(\mathcal{A}) r(|\mathcal{A} \cap \mathcal{W}|) \quad (120)$$

PAV is known [38, 43] to be equivalent to D'Hondt under closed lists. Aziz *et al.* [3] proved that PAV is NP-complete.

Aziz *et al.* [2] proved that PAV satisfies EJR. Therefore,

LEMMA 10 *PAV satisfies the lower quota.*

We finish this section showing the outputs produced by ODH, OODH, PAV, and RAV in five different elections σ^{d1} , σ^{d2} , σ^{e1} , σ^{e2} , and σ^f . The set of candidates in all elections is $\mathcal{C} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The number of seats for elections σ^{d1} and σ^{e1} is 1, and for the others is 2. The value of B for each election is shown in Table 8.

The set of winners (when ties happen all the possible sets of winners are given) for each election and voting rule are shown in Table 9.

Elections σ^{d2} and σ^{e2} are examples in which the output of ODH and OODH are different. It is easy to see that

Candidates	B (σ^{d1} and σ^{d2})	B (σ^{e1} and σ^{e2})	B (σ^f)
{a, b}	3	3	4
{b, c}	3	3	0
{a}	3	3	6
{b}	1	2	2
{c}	3	3	5

Table 8: Result of elections σ^{d1} , σ^{d2} , σ^{e1} , σ^{e2} , and σ^f

Voting rule	Election				
	σ^{d1}	σ^{d2}	σ^{e1}	σ^{e2}	σ^f
ODH	{b}	{a, b} {b, c}	{b}	{a, b} {b, c}	{a, b}
OODH	{b}	{a, c}	{b}	{a, c}	{a, b}
RAV	{b}	{a, b} {b, c}	{b}	{a, b} {b, c}	{a, c}
PAV	{b}	{a, c}	{b}	{a, b} {b, c}	{a, c}

Table 9: Sets of winners with ODH, OODH, RAV, and PAV for elections σ^{d1} , σ^{d2} , σ^{e1} , σ^{e2} , and σ^f

$$\begin{aligned}
\max\text{Min}(\sigma^{d2}, \{\mathbf{a}, \mathbf{c}\}) &= 6 > \max\text{Min}(\sigma^{d2}, \{\mathbf{a}, \mathbf{b}\}) \\
&= \max\text{Min}(\sigma^{d2}, \{\mathbf{b}, \mathbf{c}\}) = 5 \\
\max\text{Min}(\sigma^{e2}, \{\mathbf{a}, \mathbf{c}\}) &= 6 > \max\text{Min}(\sigma^{e2}, \{\mathbf{a}, \mathbf{b}\}) \\
&= \max\text{Min}(\sigma^{e2}, \{\mathbf{b}, \mathbf{c}\}) = 5, 5
\end{aligned}$$

The output of OODH for elections σ^{d1} and σ^{d2} , and elections σ^{e1} and σ^{e2} shows that OODH is not house monotonic. Also, the output of PAV for elections σ^{d1} and σ^{d2} shows that PAV is not house monotonic.

THEOREM 11 *OODH and PAV fail house monotonicity.*

8.2.1 Discussion

The analysis of PAV and RAV shows the advantages of ODH over other voting rules when representation of the different opinions or preferences of the agents is desirable. RAV fails lower quota and even JR. This makes possible that

a majority of the agents voting strategically can leave a minority without representation even if such minority is big enough to deserve one or several seats.

On the other hand, PAV exhibits a performance with respect to representation similar to ODH, but it has the crucial disadvantage that it is intractable. Voting rules that cannot be computed in polynomial time are interesting from a theoretical point of view. However, as discussed by Bartholdi III [5], voting rules that cannot be computed in polynomial time are mostly useless in practice: they can be used only in very small elections.

Another difference between ODH and PAV, as shown by the examples presented in Tables 8 and 9, is that ODH (and RAV) guarantees that the most approved candidate is always elected (in case that several candidates are tied as the most approved, ODH and RAV guarantee that at least one of them is elected). This is not the case for PAV (and also for OODH).

8.3 Other approval-based multi-winner voting rules

First of all, we review known results for the following voting rules, also surveyed by Kilgour [31].

Approval Voting (AV). Under AV, the winners are the S candidates that receive the largest number of votes. Formally, for each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, the approval score of a candidate c is $\sum_{\mathcal{A} \subseteq \mathcal{C}: c \in \mathcal{A}} B(\mathcal{A})$. The S candidates with higher approval scores are chosen.

Satisfaction Approval Voting (SAV). An agent's *satisfaction score* is the fraction of her approved candidates that are elected. SAV maximizes the sum of the agents' satisfaction scores. Formally, for each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$:

$$\text{SAV}(\sigma) = \underset{\mathcal{W} \subseteq \mathcal{C}: |\mathcal{W}|=S}{\operatorname{argmax}} \sum_{\mathcal{A} \subseteq \mathcal{C}} B(\mathcal{A}) \frac{|\mathcal{A} \cap \mathcal{W}|}{|\mathcal{A}|} \quad (121)$$

Minimax Approval Voting (MAV). MAV selects the set of candidates \mathcal{W} that minimizes the maximum *Hamming distance* [27] between \mathcal{W} and the agents' ballots. In the context of MAV, let $d(\mathcal{A}, \mathcal{B}) = |\mathcal{A} - \mathcal{B}| + |\mathcal{B} - \mathcal{A}|$, for each pair of candidate subsets \mathcal{A} and \mathcal{B} . Then, for each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$:

$$\text{MAV}(\sigma) = \underset{\mathcal{W} \subseteq \mathcal{C}: |\mathcal{W}|=S}{\operatorname{argmin}} \left(\max_{\mathcal{A} \subseteq \mathcal{C}: B(\mathcal{A}) > 0} d(\mathcal{W}, \mathcal{A}) \right) \quad (122)$$

It is straightforward to show that AV and SAV can be computed in polynomial time [3]. However, LeGrand et al. show in [32] that computing the

set of winners in MAV is NP-hard.

Aziz *et al.* proved in [2] that AV, SAV, and MAV fail JR, and therefore,

LEMMA 11 *AV, SAV, and MAV fail the lower quota.*

8.3.1 The Chamberlin and Courant and Monroe voting rules

Chamberlin and Courant [12] proposed a procedure for electing committees based on a measure of its representativeness for each agent involved in the election. For a given committee its measure of representativeness is computed as follows. Each agent is assumed to be represented by the member of the committee who ranks higher in her preference order. Then, the overall representativeness of the committee is measured by aggregating the representativeness values of such committee for each agent. Then, the elected committee should be the one with highest representativeness.

Several variants of this procedure can be proposed, depending on how the missrepresentation values for each agent are computed, and on how the missrepresentation values of the agents are aggregated. In the case of approval-based multi-winner elections, the usual missrepresentation values are one when an agent is represented by a member of the committee that she does not approve, and zero otherwise.

Two different ways of aggregating missrepresentation values are considered. The approach proposed by Harsanyi [28] (also known as the utilitarian version) consists of computing the sum of the missrepresentation values for each agent. An alternative approach to aggregate missrepresentation values was proposed by Rawls [40]. Rawls proposes a missrepresentation aggregation function that returns the maximum of the missrepresentation values for each agent. This is also known as the egalitarian version.

In summary, there are two variants of the Chamberlin and Courant voting rule for approval-based multi-winner elections. The Chamberlin–Courant–Harsanyi–approval voting rule (abbreviated CCHA), can be modelled as follows. For each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$:

$$\text{CCHA}(\sigma) = \underset{W \subseteq \mathcal{C}: |W|=S}{\operatorname{argmin}} \sum_{\mathcal{A} \subseteq \mathcal{C}: \mathcal{A} \cap W = \emptyset} B(\mathcal{A}) \quad (123)$$

In the case of the Chamberlin–Courant–Rawls–approval (CCRA), there are only two possible missrepresentation values. When sets of winners such that all agents are represented by a candidate they approve exist the missrepresentation value is 0. Otherwise the missrepresentation value is 1 and any subset of S candidates is tied as winner.

For CCRA, the sets of winners when the misrepresentation value is 0 can be formally modeled as follows:

$$\text{CCRA}_0(\sigma) = \{\mathcal{W} \subseteq \mathcal{C} : |\mathcal{W}| = S \text{ and } \sum_{\mathcal{A} \subseteq \mathcal{C} : \mathcal{A} \cap \mathcal{W} = \emptyset} B(\mathcal{A}) = 0\} \quad (124)$$

Then, CCRA is defined as:

$$\begin{aligned} \text{CCRA}(\sigma) = & \quad \text{if } \text{CCRA}_0(\sigma) = \emptyset \\ & \text{then } \{\mathcal{W} \subseteq \mathcal{C} : |\mathcal{W}| = S\} \\ & \text{else } \text{CCRA}_0(\sigma) \end{aligned} \quad (125)$$

A major drawback of the approach of Chamberlin and Courant is that the number of agents represented by each elected candidate can be very different. To remedy this, Chamberlin and Courant suggested to use weighted votes in the assembly where the vote of each elected candidate has a weight equal to the number of agents she represents. This approach has a number of problems. First of all, this approach is not applicable if the goal of the election is not to choose representatives. Secondly, it breaks the usual rule in an assembly 'one member of assembly one vote'. This was criticized by Monroe [35] that proposed instead that each candidate of the committee should represent at least $\lfloor |\mathcal{V}|/S \rfloor$ and at most $\lceil |\mathcal{V}|/S \rceil$ agents.

For each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, and each candidate subset $\mathcal{W} \subseteq \mathcal{C}, |\mathcal{W}| = S$, we define the Monroe's agent to candidate assignment functions family $\mathfrak{M}_{\sigma, \mathcal{W}}$ as the set of all functions M from $2^{\mathcal{C}} \times \mathcal{W}$ to \mathbb{N} that satisfy the following conditions.

$$\sum_{c \in \mathcal{W}} M(\mathcal{A}, c) = B(\mathcal{A}) \quad \text{for each } \mathcal{A} \subseteq \mathcal{C} \quad (126)$$

$$\lfloor \frac{|\mathcal{V}|}{S} \rfloor \leq \sum_{\mathcal{A} \in 2^{\mathcal{C}}} M(\mathcal{A}, c) \leq \lceil \frac{|\mathcal{V}|}{S} \rceil \quad \text{for each } c \in \mathcal{W} \quad (127)$$

$M(\mathcal{A}, c)$ is the number of agents that approve only the candidates in \mathcal{A} that, according to M , are represented by candidate $c \in \mathcal{W}$.

Then, the Monroe–Harsanyi–approval voting rule (MHA) is formally defined as follows. For each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$,

$$\text{MHA}(\sigma) = \underset{\mathcal{W} \subseteq \mathcal{C} : |\mathcal{W}| = S}{\operatorname{argmin}} \left(\min_{M \in \mathfrak{M}_{\sigma, \mathcal{W}}} \sum_{\mathcal{A} \subseteq \mathcal{C}} \sum_{c \in \mathcal{W} : c \notin \mathcal{A}} M(\mathcal{A}, c) \right). \quad (128)$$

Finally, as in the case of CCRA, for the Monroe–Rawls–approval (MRA) voting rule there are only two possible missrepresentation values: 0 (when all agents are represented by a candidate they approve) and 1.

For each approval-based multi-winner election $\sigma = \langle \mathcal{V}, \mathcal{C}, S, B \rangle$, the sets of winners that with MRA have a missrepresentation value of 0 are:

$$\begin{aligned} \text{MRA}_0(\sigma) = & \{ \mathcal{W} \subseteq \mathcal{C} : |\mathcal{W}| = S \text{ and} \\ & \text{exists } M \in \mathfrak{M}_{\sigma, \mathcal{W}} \text{ such that} \\ & \sum_{\mathcal{A} \in 2^{\mathcal{C}}} \sum_{c \in \mathcal{W}, c \notin \mathcal{A}} M(\mathcal{A}, c) = 0 \} \end{aligned} \quad (129)$$

Finally, the definition of MRA is:

$$\begin{aligned} \text{MRA}(\sigma) = & \quad \text{if } \text{MRA}_0(\sigma) = \emptyset \\ & \text{then } \{ \mathcal{W} \subseteq \mathcal{C} : |\mathcal{W}| = S \} \\ & \text{else } \text{MRA}_0(\sigma) \end{aligned} \quad (130)$$

All these voting rules are NP-complete: Procaccia et al. [39] proved the NP-completeness of CCHA and MHA, while Betzler et al. [8] proved that CCRA and MRA are also NP-complete.

With respect to the social choice properties, the Chamberlin–Courant variants fail house monotonicity.

THEOREM 12 *The Chamberlin–Courant–Harsanyi–approval voting rule and the Chamberlin–Courant–Rawls–approval voting rule fail house monotonicity.*

Proof.

Consider election σ^{g1} , where the set of candidates is $\mathcal{C} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The number of seats is $S = 1$. 10 agents cast the following votes: 3 agents approve $\{\mathbf{a}, \mathbf{b}\}$, 3 agents approve $\{\mathbf{a}, \mathbf{c}\}$, 2 agents approve $\{\mathbf{b}\}$, and 2 agents approve $\{\mathbf{c}\}$. Consider also election σ^{g2} . All components of election σ^{g2} are equal to those of election σ^{g1} except the number of seats that for election σ^{g2} is $S = 2$.

The set of winners with CCHA for election σ^{g1} is $\{\mathbf{a}\}$, because it minimizes the missrepresentation value (only 4 agents do not approve \mathbf{a}). The set of winners with CCHA for election σ^{g2} is $\{\mathbf{b}, \mathbf{c}\}$, because all the agents approve at least one candidate in the set of winners, and therefore the missrepresentation value is 0.

There are three sets of winners tied with CCRA for election σ^{g1} : $\{\mathbf{a}\}$, $\{\mathbf{b}\}$, and $\{\mathbf{c}\}$, because no candidate is approved by all the agents. There is

only one possible set of winners with CCRA for election σ^{g2} : $\{\mathbf{b}, \mathbf{c}\}$, because it is the only possible set of winners such that all the agents approve at least one candidate in the set of winners.

Therefore, depending on how ties are broken, it is possible that the set of winners for CCRA is $\{\mathbf{a}\}$ for election σ^{g1} but candidate \mathbf{a} cannot be a winner for election σ^{g2} . \square

The Monroe voting rules fail the lower quota.

THEOREM 13 *The Monroe–Harsanyi–approval voting rule and the Monroe–Rawls–approval voting rule fail the lower quota.*

Proof.

Consider election σ^h . The set of candidates is $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\}$. The number of seats is $S = 7$. 10 agents cast the following votes: 6 agents approve $\{c_1, c_2, c_3, c_4\}$, 1 agent approves $\{c_5\}$, 1 agent approves $\{c_6\}$, 1 agent approves $\{c_7\}$, and 1 agent approves $\{c_8\}$. The lower quota requires that $\lfloor \frac{B(\{c_1, c_2, c_3, c_4\})}{|\mathcal{V}|} S \rfloor = \lfloor \frac{6}{10} 7 \rfloor = \lfloor 4.2 \rfloor = 4$ candidates from $\{c_1, c_2, c_3, c_4\}$ (that is, all of them) have to be elected.

For this election, the Monroe voting rules require that 3 of the winners represent 2 agents and that the 4 remaining winners represent 1 agent. It is then possible both with MHA and with MRA to achieve a misrepresentation value of 0 if 4 of the winners are c_5, c_6, c_7 , and c_8 (each of these winners represents only the agent that approves her) and the other 3 winners are any from $\{c_1, c_2, c_3, c_4\}$ (each of these winners represents two of the agents that have casted $\{c_1, c_2, c_3, c_4\}$). Any other possible set of winners has a misrepresentation value strictly greater than 0 both with MHA and MRA. \square

Table 10 summarizes the results that we have found for the different voting rules that have been analyzed.

9 Conclusions

In this paper we have presented two voting rules that extend the D’Hondt method for approval-based multi-winner elections. Both voting rules are based on the same principle: to maximize the support of the least supported winner. The difference between these voting rules is that one of them (OODH) makes a global optimization while the other (ODH) follows an iterative process in which local optimizations are made at each iteration. OODH outputs sometimes sets of winners that are better than those outputted by ODH with respect to the maximum support for the least supported winner. However, we have also seen that OODH is computationally intractable

Voting rule	Complexity	Lower quota	House Monot.	Population Monot.
ODH	P	☺	☺	☺
OODH	NP-hard	☹	☹	☺
RAV	P	☹		
PAV	NP-complete	☺	☹	
AV	P	☹		
SAV	P	☹		
MAV	NP-hard	☹		
CCHA	NP-complete		☹	
CCRA	NP-complete		☹	
MHA	NP-complete	☹		
MRA	NP-complete	☹		

Table 10: Properties of approval-based multi-winner voting rules

while ODH can be computed in polynomial time. Moreover, we have proved that ODH is house monotonic, satisfies the lower quota, and is population monotonic.

In the context of voting systems in general, several research studies exist (probably the most famous is the Arrow’s Theorem [1]) that state that it is not possible to develop a ”perfect” voting system. However, there must be no doubt about the interest of voting rules with a good set of social choice properties and that can be computed efficiently. We believe that this is the strongest point of ODH.

Multi-winner voting rules are applied often in scenarios in which it is desirable that the set of winners represents the different opinions or preferences of the agents involved in the election. This question has been studied by Aziz *et al.* in [2] and Sánchez Fernández *et al.* in [41] that have proposed axioms (respectively, the extended justified representation and the lower quota/proportional justified representation) that should be satisfied by representative voting rules. Unfortunately, the only voting rule that was known so far that satisfies these axioms is PAV and it is known that it is computationally intractable. Both Aziz *et al.* [2] and Sánchez Fernández *et al.* [41] stress the importance of finding an approval-based multi-winner voting rule that satisfies these axioms and can be computed in polynomial time. Whether such voting rule existed was not known. This issue has been solved in this paper for the lower quota/proportional justified representation

with the development of ODH.⁴ In addition to the practical interest of ODH, from a theoretical point of view, the existence of ODH proves that there exist approval-based multi-winner voting rules that satisfy the lower quota and can be computed in polynomial time. Whether a voting rule exists that satisfies the extended justified representation and can be computed in polynomial time remains an open issue.

The study we have done of other approval-based multi-winner voting rules has lead also to several interesting results. It is worth to mention that, to our knowledge, this is the first time that it is proved that the approval-based versions of the Monroe voting rule fail the lower quota (for instance, these voting rules were not analyzed in [2]).

Possible examples of the application of the ODH include elections in organizations where the lack of political structures makes it unfeasible to force candidates to organize themselves into lists, or multi-agent systems where software agents must make social choices. Another example of the application of the ODH is in group decision support systems. Group decision support systems rarely consider proportional electoral formulae, although the representation of minority interests is considered to be a positive feature in many cases [26]. One explanation for this situation may be the lack of a proportional voting system that can be computed efficiently and with a simple and open ballot structure.

In this study, we focused on the design of the ODH and theoretical analysis of its properties. The future development of the ODH should address practical aspects that we have not considered, such as the practical development of efficient ODH implementations. Several ideas can be outlined regarding this future development. First, the characteristics of the specific linear programming problems that must be solved for the ODH could be exploited to increase the efficiency of ODH implementations. For example, their structure is suitable for using generalized upper bounding techniques [14]. Second, the development of parallel ODH implementations could also be a future area of research. For instance, it is possible to develop a parallel implementation of the ODH using a MapReduce [17] architecture, where the support for each candidate can be computed in parallel in Map nodes during each outer loop iteration. Moreover, the use of parallel linear programming solvers [33] could also be considered.

From a theoretical perspective, it would also be very interesting to provide an axiomatic characterization of ODH, in the style of the works of Skowron

⁴Although ODH fails the extended justified representation, we believe that the lower quota captures well the essence of the idea of representation. Moreover, concerns with respect to the definition of the extended justified representation have been raised by Sánchez Fernández *et al.* in [41].

et al. [45] and Freeman *et al.* [23] (the case for OODH seems quite obvious: OODH is characterized by the fact that the sets of winners are those that maximize the support of the least supported winner).

With respect to OODH, the main issue that remains open is whether it is possible to define a tie-breaking rule such that OODH combined with such tie-breaking rule satisfies the lower quota.

Other possible lines of future work are an exhaustive study of the social choice properties satisfied by existing approval-based multi-winner elections and the development of an extension to the D'Hondt method for ranked ballots.

Bibliography

- [1] Kenneth J. Arrow. *Social choice and individual values*, volume 12. Yale university press, 2012.
- [2] Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. Justified representation in approval-based committee voting. In *Twenty-Ninth AAAI Conference*. AAAI Press, January 2015.
- [3] Haris Aziz, Serge Gaspers, Joachim Gudmundsson, Simon Mackenzie, Nicholas Mattei, and Toby Walsh. Computational aspects of multi-winner approval voting. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 107–115. International Foundation for Autonomous Agents and Multiagent Systems, 2015.
- [4] M. L. Balinski and H.P. Young. The quota method of apportionment. *The American Mathematical Monthly*, 82(7):701–730, 1975.
- [5] John Bartholdi III, Craig A Tovey, and Michael A Trick. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and welfare*, 6(2):157–165, 1989.
- [6] Dorothea Baumeister and Sophie Dennisen. Voter dissatisfaction in committee elections. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 1707–1708. International Foundation for Autonomous Agents and Multiagent Systems, 2015.
- [7] Mokhtar S. Bazaraa, John J. Jarvis, and Hanif D. Sherali. *Linear programming and network flows*. John Wiley & Sons, 2011.

- [8] Nadja Betzler, Arkadii Slinko, and Johannes Uhlmann. On the computation of fully proportional representation. *Journal of Artificial Intelligence Research*, 47:475–519, 2013.
- [9] Robert E. Bixby, John W. Gregory, Irving J. Lusting, Roy E. Marsten, and David F. Shanno. Very large-scale linear programming: A case study in combining interior point and simplex methods. *Operations Research*, 40(5):885–897, 1992.
- [10] Steven J Brams, D Marc Kilgour, and M Remzi Sanver. A minimax procedure for electing committees. *Public Choice*, 132(3-4):401–420, 2007.
- [11] Ioannis Caragiannis, Christos Kaklamanis, Nikos Karanikolas, and Ariel D Procaccia. Socially desirable approximations for Dodgson’s voting rule. *ACM Transactions on Algorithms (TALG)*, 10(2):6, 2014.
- [12] John R. Chamberlin and Paul N. Courant. Deliberations and representative decisions: Proportional representation and the Borda rule. *The American Political Science Review*, 77(3):718–733, 1983.
- [13] Vincent Conitzer. Making decisions based on the preferences of multiple agents. *Communications of the ACM*, 53(3):84–94, 2010.
- [14] George B. Dantzig and R. M. Van Slyke. Generalized upper bounding techniques. *Journal of Computer and System Sciences*, 1(3):213–226, 1967.
- [15] George B. Dantzig and Mukund N. Thapa. *Linear programming. Introduction*. Springer, 1997.
- [16] George B. Dantzig and Mukund N. Thapa. *Linear programming. Theory and extensions*. Springer, 2003.
- [17] Jeffrey Dean and Sanjay Ghemawat. MapReduce: simplified data processing on large clusters. *Communications of the ACM*, 51(1):107–113, 2008.
- [18] P. G. di Cortona, C. Manzi, A. Pennisi, F. Ricca, and B. Simeone. *Evaluation and Optimization of Electoral Systems*. Society for Industrial and Applied Mathematics, SIAM, 1999.
- [19] Edith Elkind, Jérôme Lang, and Abdallah Saffidine. Choosing collectively optimal sets of alternatives based on the Condorcet criterion. In *Twenty-Second International Joint Conference on Artificial Intelligence*, pages 186–191. AAAI Press, July 2011.

- [20] Edith Elkind, Piotr Faliszewski, Piotr Skowron, and Arkadii Slinko. Properties of multiwinner voting rules. In *International Conference on Autonomous Agents and Multiagent Systems*, pages 53–60. International Foundation for Autonomous Agents and Multiagent Systems, May 2014.
- [21] D. M. Farrell. *Electoral Systems. A Comparative Introduction*. Palgrave Macmillan, second edition, 2011. ISBN 978-0-230-54678-3.
- [22] Joel Franklin. Convergence in Karmarkar’s algorithm for linear programming. *SIAM Journal on Numerical Analysis*, 24(4):928–945, 1987.
- [23] Rupert Freeman, Markus Brill, and Vincent Conitzer. On the axiomatic characterization of runoff voting rules. In *Twenty-Eighth AAAI Conference on Artificial Intelligence (2014)*, 2014.
- [24] Michael Gallagher. Proportionality, disproportionality and electoral systems. *Electoral Studies*, 10(1):33–51, 1991.
- [25] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Series of books in the mathematical sciences. W. H. Freeman, 1979. URL <https://books.google.es/books?id=fjxGAQAAIAAJ>.
- [26] Bezalel Gavish and Jr. John H. Gerdes. Voting mechanisms and their implications in a GDSS environment. *Annals of Operations Research*, 71:41–74, 1997.
- [27] Richard W. Hamming. Error detecting and error correcting codes. *Bell System technical journal*, 29(2):147–160, 1950.
- [28] John C. Harsanyi. Can the maximin principle serve as a basis for morality? a critique of John Rawls’s theory. *American Political Science Review*, 69(2):594–606, 1975.
- [29] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4:373–395, 1984.
- [30] L.G. Khachian. A polynomial algorithm for linear programming. *Doklady Akademii Nauk USSR*, 244:1093–1096, 1979.
- [31] D. Marc. Kilgour. Approval balloting for multi-winner elections. In Jean-Francois Laslier and M. Remzi Sanver, editors, *Handbook on Approval Voting*, pages 105–124. Springer, 2010.

- [32] Rob LeGrand, Evangelos Markakis, and Aranyak Mehta. Approval voting: local search heuristics and approximation algorithms for the min-max solution. In *First International Workshop on Computational Social Choice (COMSOC 2006)*, pages 276–289, Amsterdam, Netherlands, 2006.
- [33] Irvin J. Lusting and Edward Rothberg. Gigaflops in linear programming. *Operations Research Letters*, 18(4):157–165, 1996.
- [34] N. Megiddo. On the complexity of linear programming. In *Advances in economic theory: Fifth world congress*, pages 225–268. Cambridge University Press, 1987.
- [35] Burt L. Monroe. Fully proportional representation. *The American Political Science Review*, 89(4):925–940, 1995.
- [36] H. Nurmi. *Comparing Voting Systems*. D. Reidel Publishing Company, 1987.
- [37] Christos H. Papadimitriou and Kenneth Steiglitz. *Combinatorial Optimization. Algorithms and Complexity*. Dover Publications, 1998.
- [38] Enric Plaza. Technologies for political representation and accountability. In *EU-LAT Workshop on e-Government and e-Democracy. Santiago, Chili*, pages 99–107, 2004.
- [39] Ariel D. Procaccia, Jeffrey S. Rosenschein, and Aviv Zohar. On the complexity of achieving proportional representation. *Social Choice and Welfare*, 30(3):353–362, 2008.
- [40] John Rawls. *A theory of justice*. Harvard university press, revised edition, 1999.
- [41] Luis Sánchez-Fernández, Norberto Fernández-García, Jesús A. Fisteus, and Pablo Basanta. Some notes on justified representation. In *Tenth Multidisciplinary Workshop on Advances in Preference Handling (MPREF 2016)*, New York, USA, 2016.
- [42] T. W. Sandholm. Distributed rational decision making. In G. Weiss, editor, *Multiagent Systems. A Modern Approach to Distributed Artificial Intelligence*, pages 201–258. The MIT Press, 1999. ISBN 0-262-23203-0.
- [43] Forest Simmons. Proportional representation via approval voting. <https://www.mail-archive.com/election-methods-list@eskimo.com/msg04820.html>, 2001. [Online; accessed 25-June-2016].

- [44] Piotr Skowron, Piotr Faliszewski, and Arkadii Slinko. Fully proportional representation as resource allocation: Approximability results. In *Twenty-Third International Joint Conference on Artificial Intelligence*, pages 353–359. AAAI Press, August 2013.
- [45] Piotr Skowron, Piotr Faliszewski, and Arkadii Slinko. Axiomatic characterization of committee scoring rules. In *Sixth International Workshop on Computational Social Choice (2016)*, 2016.
- [46] D. R. Woodall. Properties of preferential election rules. *Voting matters*, 3, 1994.